

SYMBOLIC BLENDER-HORSESHOES AND APPLICATIONS

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ABSTRACT. We study partially-hyperbolic skew-product maps over the Bernoulli shift with Hölder dependence on the base points. In the case of contracting fiber maps, symbolic blender-horseshoe is defined as an invariant set which meets any almost horizontal disk in a robust sense. These invariant sets are understood as blenders with center stable bundle of any dimension. We then give necessary conditions (covering property) on an iterated function system such that the relevant skew-product has a symbolic blender-horseshoe. We use this local plug to yield robustly non-hyperbolic transitive diffeomorphisms and robust heterodimensional cycles of co-index equal to the dimension of the central direction.

1. INTRODUCTION

Currently hyperbolic systems are quite well understood from the topological and statistical perspectives, see [BDV05]. A natural objective is to characterize obstructions to hyperbolicity. In [Pal00], Palis conjectured that generation of cycles (heterodimensional cycles and homoclinic tangencies) is the only obstruction: every diffeomorphism can be approximated either by a hyperbolic one or by one with a cycle. This conjecture was proved for surface C^1 -diffeomorphisms in [PS00] and in higher dimensions there are some partial results, see [CP10].

On the other hand, dynamical properties which exhibit some “persistence” play an important role in dynamics. A property of a diffeomorphism is robust if there is a neighborhood consisting of diffeomorphisms satisfying such a property. Bearing in mind this concept, for the C^1 -case Bonatti formulated in [Bon04] a strong version of “hyperbolicity versus cycles” conjecture: the set of hyperbolic diffeomorphisms and the set of diffeomorphisms with robust cycles are two open sets whose union is dense in the space of C^1 -diffeomorphisms. This conjecture was proved in [BD08] for the so-called “tame systems”.

In this paper we study the generation of C^1 -robust cycles continuing the work of [BD08, BD12]. One of the main tools used are the so-called “blenders”, which are hyperbolic sets in some sense similar to the thick horseshoes introduced by Newhouse [New70] in the setting of homoclinic tangencies. Roughly, a blender is a local plug that guarantees the closure of an

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invariant manifold of a hyperbolic set to be greater than the dimension of its unstable bundle.

Besides playing a key role in the generation of robust cycles (see [BD08, BD12]), blenders are also important in other settings such as construction of non-hyperbolic robustly transitive sets [BD96], discontinuity of the dimension of hyperbolic sets [BDV95], stable ergodicity [RHRHTU07], and diffusion in symplectic and Hamiltonian settings [NP12].

The papers [BD96, BD08, BD12] consider blenders having a one-dimensional “central” direction, see Definition 1.1. Here, following the ideas in [NP12] we study blenders with “central” direction of any dimension, which enables us to consider these sets in more general settings.

Blenders appear in a natural way in the context of skew-product diffeomorphisms

$$f : [-1, 1]^n \times [-1, 1] \rightarrow \mathbb{R}^n \times \mathbb{R}, \quad f(x, y) = (F(x), \phi(x, y)),$$

where $n \geq 2$, $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has a horseshoe $\Lambda \subset [-1, 1]^n$, and the fiber maps $\phi(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ are contractions. Since $F|_\Lambda$ is conjugate to a shift, this map can be studied symbolically. This leads to consider *symbolic skew-product maps* Φ over the shift map $\tau : \Sigma_k = \{1, \dots, k\}^{\mathbb{Z}} \rightarrow \Sigma_k$ of $k \geq 2$ symbols of the form

$$\Phi : \Sigma_k \times M \rightarrow \Sigma_k \times M, \quad \Phi(\xi, x) = (\tau(\xi), \phi_\xi(x)), \quad (1.1)$$

where M is a manifold and $\phi_\xi : M \rightarrow M$ are diffeomorphisms depending continuously with respect to ξ . The set Σ_k is called *base* and M is the *fiber*. In this paper we will work with maps of the form (1.1) and in order to emphasize the role of the fiber maps we write $\Phi = \tau \ltimes \phi_\xi$.

1.1. Symbolic blender-horseshoes. Before introducing symbolic blender-horseshoes let us recall the definition of a blender (with one-dimensional center contracting direction) in [BDV05, BD12]:

Definition 1.1 (*cs-blenders*). *Let f be a C^1 -diffeomorphism of compact manifold M and $\Gamma \subset M$ a transitive hyperbolic set of f with a dominated splitting of the form $E^{ss} \oplus E^{cs} \oplus E^u$, where its stable bundle $E^s = E^{ss} \oplus E^{cs}$ has dimension equal to $k \geq 2$ and E^{cs} is one-dimensional. The set Γ is a cs-blender if it has a C^1 -robust superposition region of embedding disks \mathcal{H} , that is, there are a C^1 -neighborhood \mathcal{V} of f and a C^1 -open set \mathcal{H} of embeddings of $k-1$ dimensional disks H into M such that for every diffeomorphism $g \in \mathcal{V}$, every disk $H \in \mathcal{H}$ intersects the local unstable manifold $W_{loc}^u(\Gamma_g)$ of the continuation Γ_g of Γ for g .*

The definition of symbolic blender-horseshoe involves a set $\mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ of symbolic skew-product maps of the form (1.1), where D is a bounded open set of M , k is the number of symbols of the shift map, and $0 \leq \alpha \leq 1$, $0 < \lambda < \beta$ refer to the Hölder exponent and the contraction/expansion constants of the fiber maps, respectively. See the precise definitions in Section 1.5.

To adapt the definition of a blender to the symbolic context we need to define the family of almost horizontal disks, which provides the superposition region.

Definition 1.2 (Almost horizontal disks). *Consider $\mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$. Given $\delta > 0$ and an open set $B \subset D$, we say that $H^s \subset \Sigma_k \times M$ is an almost δ -horizontal disk in $\Sigma_k \times B$ if there are $(\zeta, z) \in \Sigma_k \times B$ and a (α, C) -Hölder function*

$$h : W_{loc}^s(\zeta; \tau) \rightarrow B \text{ such that } \|z - h(\xi)\| < \delta \text{ for all } \xi \in W_{loc}^s(\zeta; \tau),$$

with

$$C\nu^\alpha < \delta \quad \text{and} \quad H^s = \{(\xi, h(\xi)) : \xi \in W_{loc}^s(\zeta; \tau)\}.$$

Here $W_{loc}^s(\zeta; \tau) \stackrel{\text{def}}{=} \{\xi \in \Sigma_k : \xi_i = \zeta_i, i \geq 0\}$ is the local stable set of ζ for τ and $0 < \nu < 1$ such that $d_{\Sigma_k}(\tau(\xi), \tau(\zeta)) = \nu$ for all $\xi \in W_{loc}^s(\zeta; \tau)$.

We are now ready to formulate the definition of a symbolic blender.

Definition 1.3 (Symbolic cs-blender-horseshoe). *Consider a symbolic skew-product map $\Phi \in \mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ with $\beta < 1$ and $\alpha > 0$.*

The maximal invariant set Γ_Φ of Φ in $\Sigma_k \times \overline{D}$ is a symbolic cs-blender-horseshoe if there are $\delta > 0$, a non-empty open set $B \subset D$, and a neighborhood \mathcal{V} of Φ in $\mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ such that for every $\Psi \in \mathcal{V}$ and every δ -horizontal disk H^s in $\Sigma_k \times B$ it holds that ^{}*

$$W_{loc}^{uu}(\Gamma_\Psi; \Psi) \cap H^s \neq \emptyset,$$

where Γ_Ψ is the maximal invariant set of Ψ in $\Sigma_k \times \overline{D}$.

The family of δ -horizontal disks in $\Sigma_k \times B$ is called the *superposition region* of the symbolic blenders and the open set B is the *superposition domain*.

1.2. Iterated function systems, and generation of blenders. A special case of symbolic skew-product maps are the *one-step* ones where the fiber maps ϕ_ξ only depend on the coordinate ξ_0 of $\xi = (\xi_i)_{i \in \mathbb{Z}} \in \Sigma_k$. So, we have that $\phi_\xi = \phi_i$ if $\xi_0 = i$, and write $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$. In this paper, blender-horseshoes are obtained by small perturbations in $\mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ of one-step skew-product maps.

Blenders in the one-step setting (see Definition 3.1) were first introduced in [NP12] where the fiber maps satisfy the *covering* and *well-distribution of periodic points* properties. Our construction does not use the well-distribution property (compare with [NP12, Proposition 3.6]), but involves the so-called Hutchinson theory for iterated function systems [Hut81], defined as follows.

Given maps $\phi_1, \dots, \phi_k : \overline{D} \rightarrow D$, we denote by $\text{IFS}(\phi_1, \dots, \phi_k)$, the set of all compositions of the maps ϕ_1, \dots, ϕ_k and will refer to this as the associated *iterated function system* (or IFS) of the one-step map $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$. The $\text{IFS}(\Phi) \stackrel{\text{def}}{=} \text{IFS}(\phi_1, \dots, \phi_k)$ has the *covering property* if there is an open set $B \subset D$ such that

$$\overline{B} \subset \phi_1(B) \cup \dots \cup \phi_k(B).$$

One objective of this paper is to understand how do dynamical properties of $\text{IFS}(\Phi)$ translate to robust properties of $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ under perturbations in $\mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$. Our main result describes the relation between the covering property and the existence of blenders.

^{*}The exact definition of the local strong unstable set of Γ_Ψ is given in Section 2. By Theorem 2.9, it follows that $W_{loc}^{uu}(\Gamma_\Psi; \Psi) = \Gamma_\Psi$ and $\Psi|_{\Gamma_\Psi}$ is conjugated to $\tau|_{\Sigma_k}$ for all $\Psi \in \mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ with $\beta < 1$ and $\alpha > 0$.

Theorem A. Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ with $\alpha > 0$ and $\nu^\alpha < \lambda < \beta < 1$. Assume that there exists an open set $B \subset D$ such that

$$\overline{B} \subset \phi_1(B) \cup \dots \cup \phi_k(B).$$

Then the maximal invariant set Γ_Φ of Φ in $\Sigma_k \times \overline{D}$ is a symbolic cs-blender-horseshoe for Φ whose superposition domain contains B .

1.3. Applications: robust heterodimensional cycles and mixing sets.

Our strategy is to give conditions on the IFS which will imply that the associated one-step skew-product map has a symbolic cycle or is mixing (see Definition 5.4). Then, one studies the robustness of these properties under perturbations in the symbolic setting (see Theorems 5.5 and 5.10). Finally, the smooth realization of the one-step skew-product shows that the corresponding property is robust in the C^1 -topology (Proposition 5.2). We now state precisely our applications.

A *heterodimensional cycle* for a diffeomorphism $f \in \text{Diff}^1(M)$ between a pair of hyperbolic transitive sets Ω_1 and Ω_2 of different *indices* (dimension of their stable bundles) is defined by the relations

$$W^s(\Omega_1, f) \cap W^u(\Omega_2, f) \neq \emptyset \quad \text{and} \quad W^u(\Omega_1, f) \cap W^s(\Omega_2, f) \neq \emptyset.$$

The *co-index* is the absolute value of the difference between the indices of Ω_1, Ω_2 . We are interested in cycles that cannot be destroyed by perturbations.

Definition 1.4 (Robust heterodimensional cycle). *A diffeomorphism f has a C^1 -robust heterodimensional cycle associated with its transitive hyperbolic sets Ω_1 and Ω_2 if there is a C^1 -neighborhood \mathcal{U} of f such that every $g \in \mathcal{U}$ has a heterodimensional cycle associated with the hyperbolic continuations Ω_1^g and Ω_2^g of Ω_1 and Ω_2 , respectively.*

In [BD08] it is proved that cycles of co-index one yield C^1 -robust cycles after small perturbations. A key ingredient is the construction of blenders with one dimensional center-stable direction. Here we consider $F: N \rightarrow N$ a C^1 -diffeomorphism with a horseshoe and M a manifold of dimension $c \geq 1$. We prove that there are maps arbitrarily close to $F \times \text{id}$ having robust cycles. The following result is a generalization of [BD08, Theorem 2.4].

Theorem B. *Let $F: N \rightarrow N$ be a C^1 -diffeomorphism with a Smale horseshoe. Then there is an arc $\{f_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$ of C^1 -diffeomorphisms on $N \times M$ such that $f_0 = F \times \text{id}$ and f_ε has a C^1 -robust heterodimensional cycle of co-index c for all $0 < \varepsilon < \varepsilon_0$.*

The construction in [BD96] of persistent non-hyperbolic transitive diffeomorphisms used a “chain of blenders”, which is a connected family of blenders. Here, we provide an alternative construction with symbolic blenders, thus obtaining a slightly stronger, topologically mixing, version of the theorem. An invariant set X is *topologically mixing* for a map g if for any pair of open sets U, V of X there is a $n_0 > 0$ such that $g^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$.

Theorem C. *Let $F: N \rightarrow N$ be a C^1 -diffeomorphism with a Smale horseshoe Λ . Then there is an arc $\{f_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$ of C^1 -diffeomorphisms on $N \times M$ such that $f_0 = F \times \text{id}$ and for every small enough C^1 -perturbation g of f_ε , $0 < \varepsilon < \varepsilon_0$, there is a g -invariant set $\Delta_g \subset N \times M$ homeomorphic to $\Lambda \times M$ which is a non-hyperbolic topologically mixing set for g .*

1.4. Organization of the paper. In the next section we will study symbolic skew-products in $\mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ with $\beta < 1$, and properties of the maximal invariant set in $\Sigma_k \times \overline{D}$. In Section 3 we will look at symbolic blender-horseshoes in the one-step setting and then prove Theorem A in Section 4. Finally, in Section 5, we will prove Theorems B and C.

1.5. The set of symbolic skew-products. Given positive constants $0 < \lambda < \beta$, a map $\phi: \overline{D} \rightarrow D$ is said to be a (λ, β) -Lipschitz on \overline{D} if

$$\lambda \|x - y\| < \|\phi(x) - \phi(y)\| < \beta \|x - y\|, \quad \text{for all } x, y \in \overline{D},$$

where $\|\cdot\|$ is the metric of M and $\|x - y\|$ denotes the distance between x and y in M . For a fixed $0 \leq \alpha \leq 1$, we say that $\Phi = \tau \ltimes \phi_\xi$ is *locally α -Hölder continuous* (or that its fiber maps ϕ_ξ depend *locally α -Hölder continuously* on \overline{D} with respect to the base point ξ) if there is a constant $C \geq 0$ such that

$$\|\phi_\xi^{\pm 1}(x) - \phi_\zeta^{\pm 1}(x)\| \leq C d_{\Sigma_k}(\xi, \zeta)^\alpha \quad \text{for all } x \in \overline{D} \quad (1.2)$$

and every $\xi, \zeta \in \Sigma_k$ with $\xi_0 = \zeta_0$, where d_{Σ_k} is the metric defined by

$$d_{\Sigma_k}(\xi, \zeta) \stackrel{\text{def}}{=} \nu^\ell, \quad \ell = \min\{i \in \mathbb{Z}^+ : \xi_i \neq \zeta_i \text{ or } \xi_{-i} \neq \zeta_{-i}\}, \quad 0 < \nu < 1.$$

We denote by C_Φ the smallest non-negative constant satisfying (1.2) and call it the *local Hölder constant* of Φ on \overline{D} .

Definition 1.5 (Sets of symbolic skew-products). *Let $D \subset M$ be a bounded open set, $r \geq 0$, $0 < \lambda < \beta$, $0 \leq \alpha \leq 1$ and $k > 1$. We define $\mathcal{S}_{k,\lambda,\beta}^{r,\alpha}(D)$ as the set of symbolic skew-product maps $\Phi = \tau \ltimes \phi_\xi$ as in (1.1) such that*

- ϕ_ξ is C^r - (λ, β) -Lipschitz on \overline{D} for all $\xi \in \Sigma_k$,
- ϕ_ξ depends locally α -Hölder continuously on \overline{D} with respect to ξ ,
- if $\beta < 1$ then $\phi_\xi(\overline{D}) \subset D$ for all $\xi \in \Sigma_k$, and
- if $1 < \lambda$ then $\overline{D} \subset \phi_\xi(D)$ for all $\xi \in \Sigma_k$.

The set $\mathcal{S}_{k,\lambda,\beta}^{r,\alpha}(D)$ is endowed with the distance

$$d_S(\Phi, \Psi) = \sup_{\xi \in \Sigma_k} d_{C^r}(\phi_\xi, \psi_\xi) + |C_\Phi - C_\Psi|, \quad (1.3)$$

where $\Phi = \tau \ltimes \phi_\xi$ and $\Psi = \tau \ltimes \psi_\xi$.

1.6. Standing notation. Given a bi-sequence $\xi = (\dots, \xi_{-1}; \xi_0, \xi_1, \dots) \in \Sigma_k$ the symbol at the right of “;” is the “0 coordinate” of ξ . Given $\Phi = \tau \ltimes \phi_\xi$, for every $n > 0$ and every $(\xi, x) \in \Sigma_k \times \overline{D}$ we set

$$\begin{aligned} \phi_\xi^n(x) &\stackrel{\text{def}}{=} \phi_{\tau^{n-1}(\xi)} \circ \dots \circ \phi_\xi(x), \\ \phi_\xi^{-n}(x) &\stackrel{\text{def}}{=} \phi_{\tau^{-(n-1)}(\xi)}^{-1} \circ \dots \circ \phi_\xi^{-1}(x). \end{aligned} \quad (1.4)$$

Note that for every $n \geq 0$,

$$\Phi^n(\xi, x) = (\tau^n(\xi), \phi_\xi^n(x)) \quad \text{and} \quad \Phi^{-n}(\xi, x) = (\tau^{-n}(\xi), \phi_{\tau^{-1}(\xi)}^{-n}(x)).$$

1.7. *cu-blenders*. To define a symbolic *cu*-blender-horseshoe, firstly we need to introduce the associated inverse symbolic skew-product for $\Phi = \tau \ltimes \phi_\xi$. Given $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$, the symbolic skew-product

$$\Phi^* = \tau \ltimes \phi_\xi^* \in \mathcal{S}_{k,\beta^{-1},\lambda^{-1}}^{0,\alpha}(D),$$

where $\phi_\xi^* : M \rightarrow M$ given by $\phi_\xi^*(x) = \phi_{\tau^{-1}(\xi^*)}^{-1}(x)$, is called the *associated inverse skew-product* for Φ . Here

$$\xi^* = (\dots \xi_1; \xi_0, \xi_{-1}, \dots)$$

denotes the conjugate sequence of $\xi = (\dots \xi_{-1}; \xi_0, \xi_1, \dots)$. Note that since $\tau(\xi)^* = \tau^{-1}(\xi^*)$, then iterates of Φ^* correspond to iterates of Φ^{-1} . This observation allows us to define symbolic *cu*-blender-horseshoes for skew-products in $\mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ with $\lambda > 1$ and $\alpha > 0$. Namely, *symbolic cu-blender-horseshoe* for Φ is defined as a symbolic *cs*-blender-horseshoe for Φ^* .

For the rest of the article we just consider symbolic *cs*-blender-horseshoes, calling them for short symbolic blenders. We also fix the following quantities: the number $k \geq 2$ of symbols, the contractivity ratio $0 < \nu < 1$ of shift map $\tau : \Sigma_k \rightarrow \Sigma_k$, the Hölder exponent $0 \leq \alpha \leq 1$, the non-empty bounded open set D and the local Lipschitz constants λ and β . Thus, for simplicity we just write \mathcal{S} in the place of $\mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$.

2. SYMBOLIC SKEW-PRODUCTS

In this section given $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}$ with $0 < \lambda < \beta < 1$ and $\alpha > 0$ we study the maximal invariant set Γ_Φ of Φ in $\Sigma_k \times \overline{D}$. We will prove that $\Phi|_{\Gamma_\Phi}$ is conjugated to $\tau|_{\Sigma_k}$, $W_{loc}^{uu}(\Gamma_\Phi; \Phi) = \Gamma_\Phi$ and the projection of Γ_Φ in M depends continuously in the Hausdorff metric on Φ (see Theorem 2.9). This is a version of well-known results from [HPS77] in our setting of symbolic skew-products.

2.1. Invariant graph. The invariant graph theorem is an important tool to understand symbolic blenders. Next result claims the existence of a “unique invariant attracting graph” on $\Sigma_k \times \overline{D}$ for $\Phi \in \mathcal{S}$. Indeed these graphs depend continuously on Φ . The theorem below is a reformulation of the results in [HPS77] (see also [Sta99, Theorem 1.1] and [BHN99, Section 6] for Item (iii)).

Given a function $g : \Sigma_k \rightarrow \overline{D}$ its *graph map* is defined by

$$\text{graph}[g] : \Sigma_k \rightarrow \Sigma_k \times M, \quad \text{graph}[g](\xi) = (\xi, g(\xi))$$

and its *graph set* by $\Gamma_g \stackrel{\text{def}}{=} \text{image}(\text{graph}[g]) = \{(\xi, g(\xi)) : \xi \in \Sigma_k\} \subset \Sigma_k \times \overline{D}$.

Theorem 2.1 ([HPS77, Sta99, BHN99]). *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}$ with $\beta < 1$ and $\alpha \geq 0$. Then there exists a unique bounded continuous function $g_\Phi : \Sigma_k \rightarrow \overline{D}$ such that*

$$i) \quad \Phi(\xi, g_\Phi(\xi)) = (\tau(\xi), g_\Phi \circ \tau(\xi)), \text{ for all } \xi \in \Sigma_k, \text{ that is, } \Phi(\Gamma_{g_\Phi}) = \Gamma_{g_\Phi},$$

ii) *for every $(\xi, x) \in \Sigma_k \times \overline{D}$ and $n \geq 0$, it holds*

$$\|\phi_\xi^n(x) - g_\Phi(\tau^n(\xi))\| \leq \beta^n \|g_\Phi(\xi) - x\|,$$

iii) *the map $\mathcal{L} : \mathcal{S} \rightarrow C^0(\Sigma_k, \overline{D})$ given by $\mathcal{L}(\Phi) = g_\Phi$ is continuous.*

For notational simplicity in what follows we just write Γ_Φ in place of Γ_{g_Φ} to denote the unique contracting invariant graph set. The next proposition shows that Γ_Φ is the locally maximal invariant set in $\Sigma_k \times \overline{D}$.

Proposition 2.2. *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}$ with $\beta < 1$ and $\alpha \geq 0$. Then*

- i) the restriction $\Phi|_{\Gamma_\Phi}$ of Φ to the set Γ_Φ is conjugate to τ , and*
- ii) the graph set Γ_Φ is the (forward) maximal invariant set in $\Sigma_k \times \overline{D}$*

$$\Gamma_\Phi = \bigcap_{n \in \mathbb{Z}} \Phi^n(\Sigma_k \times \overline{D}) = \bigcap_{n \in \mathbb{N}} \Phi^n(\Sigma_k \times \overline{D}).$$

Proof. By (i) in Theorem 2.1, one has that $\Phi \circ \text{graph}[g_\Phi] = \text{graph}[g_\Phi] \circ \tau$. Hence the function $\text{graph}[g_\Phi]$ conjugates the maps $\Phi|_{\Gamma_\Phi}$ and τ . To get the continuity note that $\text{graph}[g_\Phi]$ is continuous and that $\text{graph}[g_\Phi]^{-1}: \Sigma_k \times M \rightarrow \Sigma_k$ is the projection on the first coordinate. Thus it is also continuous and we conclude Item (i) of the proposition.

Recall that periodic points of the shift map τ are dense in Σ_k , that is, $\Sigma_k = \overline{\text{Per}(\tau)}$. The conjugation in the first part of this proposition implies that

$$\Gamma_\Phi = \overline{\text{Per}(\Phi|_{\Gamma_\Phi})}.$$

Let Γ be the local maximal invariant set of Φ in $\Sigma_k \times D$,

$$\Gamma = \bigcap_{n \in \mathbb{Z}} \Phi^n(\Sigma_k \times \overline{D}),$$

then $\Gamma_\Phi \subset \Gamma$.

To prove that $\Gamma \subset \Gamma_\Phi$, given any $(\xi, x) \in \Gamma$ it is enough to see that $x = g_\Phi(\xi)$. As the set Γ_Φ is bounded, we have that

$$K = \sup\{d(\gamma, \Gamma_\Phi), \gamma \in \Gamma\} \in [0, +\infty).$$

Since the maps ϕ_ξ are contractions with contraction constant $0 < \beta < 1$ we deduce that

$$\begin{aligned} \|x - g_\Phi(\xi)\| &= \|\phi_\xi^n \circ \phi_\xi^{-n}(x) - \phi_\xi^n \circ \phi_\xi^{-n}(g_\Phi(\xi))\| \\ &\leq \beta^n \|\phi_\xi^{-n}(x) - \phi_\xi^{-n}(g_\Phi(\xi))\| \\ &= \beta^n d(\Phi^{-n}(\xi, x), \Phi^{-n}(\xi, g_\Phi(\xi))) \leq K\beta^n. \end{aligned}$$

Taking $n \rightarrow \infty$ we get $x = g_\Phi(\xi)$ and so $(\xi, x) \in \Gamma_\Phi$. Thus $\Gamma \subset \Gamma_\Phi$, proving the proposition. \square

2.2. Unstable sets. In this section we study the relation between the set Γ_Φ and the unstable sets of points $(\xi, x) \in \Gamma_\Phi$. We first show the existence of a strong unstable lamination. To this end, we define the local unstable set of ξ for τ as

$$W_{loc}^u(\xi; \tau) \stackrel{\text{def}}{=} \{\zeta \in \Sigma_k : \zeta_i = \xi_i, i < 0\}$$

and the unstable set of (ξ, x) for Φ as

$$W^u((\xi, x); \Phi) \stackrel{\text{def}}{=} \{(\zeta, y) \in \Sigma_k \times M : \lim_{n \rightarrow \infty} d(\Phi^{-n}(\zeta, y), \Phi^{-n}(\xi, x)) = 0\}.$$

Proposition 2.3. Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}$ with $\beta < 1$, $\alpha > 0$, and let Γ_Φ be the maximal invariant set of Φ in $\Sigma_k \times \overline{D}$. Then, there exists a partition

$$\mathcal{W}_{\Gamma_\Phi}^u = \{W_{loc}^{uu}((\xi, x); \Phi) : (\xi, x) \in \Gamma_\Phi\}$$

of Γ_Φ such that for every $(\xi, x) \in \Gamma_\Phi$ it holds

- i) $W_{loc}^{uu}((\xi, x); \Phi)$ is the graph of an α -Hölder map $\gamma_\xi^u : W_{loc}^u(\xi; \tau) \rightarrow M$,
- ii) $\gamma_\xi^u(\xi) = g_\Phi(\xi) = x$,
- iii) $\gamma_\xi^u(\xi'') = \gamma_{\xi'}^u(\xi'')$ for all $\xi', \xi'' \in W_{loc}^u(\xi; \tau)$,
- iv) $\phi_{\tau^{-1}(\xi')}^{-1} \circ \gamma_\xi^u(\xi') = \gamma_{\tau^{-1}(\xi)}^u \circ \tau^{-1}(\xi')$ where $\xi' \in W_{loc}^u(\xi; \tau)$, and
- v) $W_{loc}^{uu}((\xi, x); \Phi) \subset W^u((\xi, x); \Phi)$.

Proof. Let $(\xi, x) \in \Gamma_\Phi$. Note that $x = g_\Phi(\xi)$ and that $(\phi_{\tau^{-n}(\xi)}^n)^{-1} = \phi_{\tau^{-1}(\xi)}^{-n}$, indeed:

$$\phi_{\tau^{-n}(\xi)}^{-1} \circ \cdots \circ \phi_{\tau^{-1}(\xi)}^{-1} \circ \phi_{\tau^{-1}(\xi)} \circ \cdots \circ \phi_{\tau^{-n}(\xi)} = \phi_{\tau^{-1}(\xi)}^{-n} \circ \phi_{\tau^{-n}(\xi)}^n = id.$$

We define a sequence of maps $\gamma_\xi^{u,n} : W_{loc}^u(\xi; \tau) \rightarrow M$ by

$$\gamma_\xi^{u,n}(\xi') = \phi_{\tau^{-n}(\xi')}^n \circ (\phi_{\tau^{-n}(\xi)}^n)^{-1}(x) = \phi_{\tau^{-n}(\xi')}^n \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi).$$

Note that $\{\gamma_\xi^{u,n}\}$ is a sequence in the complete metric space $C^0(W_{loc}^u(\xi; \tau), M)$.

Lemma 2.4. The sequence $\{\gamma_\xi^{u,n}\}$ is Cauchy sequence and so it converges to some continuous map $\gamma_\xi^u : W_{loc}^u(\xi; \tau) \rightarrow M$.

Proof. Since $\beta, \nu < 1$, to prove the first part in the lemma it is enough to see that

$$\|\gamma_\xi^{u,n+1}(\xi') - \gamma_\xi^{u,n}(\xi')\| \leq C_\Phi(\beta\nu^\alpha)^{n+1} d_{\Sigma_k}(\xi, \xi')^\alpha. \quad (2.1)$$

To prove this inequality first recall notation in (1.4) and observe that Item (i) of Theorem 2.1 implies that for every $n > 0$ and $\xi \in \Sigma_k$

$$\phi_\xi^n \circ g_\Phi(\xi) = g_\Phi \circ \tau^n(\xi) \quad \text{and} \quad \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi) = g_\Phi \circ \tau^{-n}(\xi). \quad (2.2)$$

Since $\phi_\xi(\overline{D}) \subset D$ for all $\xi \in \Sigma_k$, then for every $0 < i \leq n$

$$\phi_{\tau^{-n}(\xi')}^{n-i} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi) = \phi_{\tau^{-n}(\xi')}^{n-i} \circ g_\Phi \circ \tau^{-n}(\xi) \in \overline{D}.$$

To prove the inequality in (2.1), observe that $\|\gamma_\xi^{u,n+1}(\xi') - \gamma_\xi^{u,n}(\xi')\|$ is equal to

$$\begin{aligned} & \|\phi_{\tau^{-n-1}(\xi')}^{n+1} \circ \phi_{\tau^{-1}(\xi)}^{-n-1} \circ g_\Phi(\xi) - \phi_{\tau^{-n}(\xi')}^n \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi)\| \\ & \leq \beta^n \|\phi_{\tau^{-n-1}(\xi')} \circ \phi_{\tau^{-n-1}(\xi)}^{-1} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi) - \\ & \quad \phi_{\tau^{-n-1}(\xi')} \circ \phi_{\tau^{-n-1}(\xi')}^{-1} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi)\| \\ & \leq \beta^{n+1} \|\phi_{\tau^{-n-1}(\xi)}^{-1} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi) - \phi_{\tau^{-n-1}(\xi')}^{-1} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi)\|. \end{aligned}$$

As $\phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi) \in \overline{D}$, then local Hölder continuity implies

$$\begin{aligned} \|\gamma_\xi^{u,n+1}(\xi') - \gamma_\xi^{u,n}(\xi')\| & \leq \beta^{n+1} C_\Phi d_{\Sigma_k}(\tau^{-n-1}(\xi), \tau^{-n-1}(\xi'))^\alpha \\ & \leq C_\Phi(\beta\nu^\alpha)^{n+1} d_{\Sigma_k}(\xi, \xi')^\alpha. \end{aligned}$$

The proof of the lemma is now complete. \square

To prove that $\mathcal{W}_{\Gamma_\Phi}^u$ is a partition of Γ_Φ we need to show that $W_{loc}^{uu}((\xi, x); \Phi)$ is contained in Γ_Φ for all $(\xi, x) \in \Gamma_\Phi$. To do this let

$$(\xi', x') = (\xi', \gamma_\xi^u(\xi')) \in W_{loc}^{uu}((\xi, x); \Phi).$$

From (2.2) and noting that $x = g_\Phi(\xi)$ we have that

$$\|g_\Phi(\xi') - \gamma_\xi^u(\xi')\| = \lim_{n \rightarrow \infty} \|\phi_{\tau^{-n}(\xi')}^n \circ g_\Phi \circ \tau^{-n}(\xi') - \phi_{\tau^{-n}(\xi')}^n \circ g_\Phi \circ \tau^{-n}(\xi)\|.$$

Since the maps ϕ_ξ are β -contractions and $g_\Phi : \Sigma_k \rightarrow \overline{D}$, if K is an upper bound of the diameter of \overline{D} we get that

$$\|g_\Phi(\xi') - \gamma_\xi^u(\xi')\| \leq \lim_{n \rightarrow \infty} \beta^n \|g_\Phi \circ \tau^{-n}(\xi') - g_\Phi \circ \tau^{-n}(\xi)\| \leq \lim_{n \rightarrow \infty} K \beta^n = 0.$$

Thus $g_\Phi(\xi') = \gamma_\xi^u(\xi') = x'$ and so $(\xi', x') \in \Gamma_\Phi$. That is,

$$W_{loc}^{uu}((\xi, x); \Phi) \subset \Gamma_\Phi \quad \text{for all } (\xi, x) \in \Gamma_\Phi.$$

Indeed, we have proved that $\gamma_\xi^u(\xi') = g_\Phi(\xi')$ for all $\xi' \in W_{loc}^u(\xi; \tau)$, proving (ii). In particular, $\gamma_\xi^u(\xi'') = g_\Phi(\xi'') = \gamma_{\xi'}^u(\xi'')$ for all $\xi', \xi'' \in W_{loc}^u(\xi; \tau)$, which proves (iii).

To prove that γ_ξ^u is an α -Hölder map (Item (i)) first note that by the triangle inequality

$$\|\gamma_\xi^{u,n}(\xi') - x\| = \|\gamma_\xi^{u,n}(\xi') - \gamma_\xi^{u,n}(\xi)\| \leq \sum_{i=1}^n s_i(\xi') \quad (2.3)$$

where $x = g_\Phi(\xi)$ and $s_i(\xi')$ is given by

$$\begin{aligned} & \|\phi_{\tau^{-n+i}(\xi')}^{n-i} \circ \phi_{\tau^{-n-1+i}(\xi')} \circ \phi_{\tau^{-n}(\xi)}^{i-1} \circ \phi_{\tau^{-1}(\xi)}^{-n}(x) - \\ & \quad \phi_{\tau^{-n+i}(\xi')}^{n-i} \circ \phi_{\tau^{-n-1+i}(\xi)} \circ \phi_{\tau^{-n}(\xi)}^{i-1} \circ \phi_{\tau^{-1}(\xi)}^{-n}(x)\|. \end{aligned}$$

Claim 2.5. $s_i(\xi') \leq C_\Phi (\beta \nu^\alpha)^{n-i} d_{\Sigma_k}(\xi, \xi')^\alpha$, for every $i \in \{1, \dots, n\}$.

We postpone the proof of this claim. Taking $n \rightarrow \infty$ in (2.3) we get

$$\|\gamma_\xi^u(\xi') - \gamma_\xi^u(\xi)\| \leq C_\gamma d_{\Sigma_k}(\xi', \xi)^\alpha \quad \text{for all } \xi' \in W_{loc}^s(\xi; \tau),$$

where $C_\gamma = C_\Phi(1 - \beta \nu^\alpha)^{-1}$, showing γ_ξ^u is α -Hölder. Indeed for $\xi', \xi'' \in W_{loc}^u(\xi; \tau)$ observe that $\gamma_\xi^u(\xi') = \gamma_{\xi'}^u(\xi')$ and $\gamma_\xi^u(\xi'') = \gamma_{\xi'}^u(\xi'')$. Thus

$$\|\gamma_\xi^u(\xi') - \gamma_\xi^u(\xi'')\| = \|\gamma_{\xi'}^u(\xi') - \gamma_{\xi'}^u(\xi'')\| \leq C_\gamma d_{\Sigma_k}(\xi', \xi'')^\alpha.$$

Note that the Hölder constant obtained is uniform on ξ and $x = g_\Phi(\xi)$.

Proof of Claim 2.5. For every $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned} s_i(\xi') &= \|\phi_{\tau^{-n+i}(\xi')}^{n-i} \circ \phi_{\tau^{-n-1+i}(\xi')} \circ \phi_{\tau^{-n}(\xi)}^{i-1} \circ \phi_{\tau^{-1}(\xi)}^{-n}(x) - \\ & \quad \phi_{\tau^{-n+i}(\xi')}^{n-i} \circ \phi_{\tau^{-n-1+i}(\xi)} \circ \phi_{\tau^{-n}(\xi)}^{i-1} \circ \phi_{\tau^{-1}(\xi)}^{-n}(x)\| \\ &\leq \beta^{n-i} \|\phi_{\tau^{-n-1+i}(\xi')} \circ \phi_{\tau^{-n}(\xi)}^{i-1} \circ \phi_{\tau^{-1}(\xi)}^{-n}(x) - \\ & \quad \phi_{\tau^{-n-1+i}(\xi)} \circ \phi_{\tau^{-n}(\xi)}^{i-1} \circ \phi_{\tau^{-1}(\xi)}^{-n}(x)\| \\ &\leq \beta^{n-i} C_\Phi d_{\Sigma_k}(\tau^{-n-1+i}(\xi'), \tau^{-n-1+i}(\xi))^\alpha \\ &\leq (\beta \nu^\alpha)^{n-i} C_\Phi d_{\Sigma_k}(\xi', \xi)^\alpha, \end{aligned}$$

where the first inequality follows from the β -contraction of ϕ_ξ , and the second one from the α -Hölder continuity of ϕ_ξ . \square

This completes the proof of Item (i).

To prove Item (iv) observe that

$$\begin{aligned}
\phi_{\tau^{-1}(\xi')}^{-1} \circ \gamma_{\xi'}^u(\xi') &= \lim_{n \rightarrow \infty} \phi_{\tau^{-1}(\xi')}^{-1} \circ \phi_{\tau^{-n}(\xi')}^n \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_{\Phi}(\xi) \\
&= \lim_{n \rightarrow \infty} \phi_{\tau^{-1}(\xi')}^{-1} \circ \phi_{\tau^{-1}(\xi')} \circ \phi_{\tau^{-n}(\xi')}^{n-1} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_{\Phi}(\xi) \\
&= \lim_{n \rightarrow \infty} \phi_{\tau^{-n}(\xi')}^{n-1} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_{\Phi}(\xi) \\
&= \lim_{n \rightarrow \infty} \phi_{\tau^{-n}(\xi')}^{n-1} \circ \phi_{\tau^{-2}(\xi)}^{-(n-1)} \circ \phi_{\tau^{-1}(\xi)}^{-1} \circ g_{\Phi}(\xi) \\
&= \lim_{n \rightarrow \infty} \phi_{\tau^{-n}(\xi')}^{n-1} \circ \phi_{\tau^{-2}(\xi)}^{-(n-1)} \circ g_{\Phi} \circ \tau^{-1}(\xi) \\
&= \gamma_{\tau^{-1}(\xi)}^u \circ \tau^{-1}(\xi'),
\end{aligned} \tag{2.4}$$

where equality (2.4) is consequence of (2.2): $\phi_{\tau^{-1}(\xi)}^{-1} \circ g_{\Phi}(\xi) = g_{\Phi} \circ \tau^{-1}(\xi)$.

Now we prove Item (v): $W_{loc}^{uu}((\xi, x); \Phi) \subset W^u((\xi, x); \Phi)$. Consider $(\xi', x') \in W_{loc}^{uu}((\xi, x); \Phi)$. Then $\xi' \in W_{loc}^u(\xi; \tau)$ and $x' = \gamma_{\xi'}^u(\xi')$. Thus,

$$\begin{aligned}
d(\Phi^{-n}(\xi', x'), \Phi^{-n}(\xi, x)) &= \\
&= d_{\Sigma_k}(\tau^{-n}(\xi'), \tau^{-n}(\xi)) + \|\phi_{\tau^{-1}(\xi')}^{-n}(x') - \phi_{\tau^{-1}(\xi)}^{-n}(x)\| \\
&\leq \nu^n d_{\Sigma_k}(\xi', \xi) + \|\phi_{\tau^{-1}(\xi')}^{-n} \circ \gamma_{\xi'}^u(\xi') - \phi_{\tau^{-1}(\xi)}^{-n}(x)\|.
\end{aligned} \tag{2.5}$$

Since $(\xi, x) \in \Gamma_{\Phi}$ we have $x = g_{\Phi}(\xi)$, and using that $\gamma_{\tau^{-n}(\xi)}^u \circ \tau^{-n}(\xi) = g_{\Phi} \circ \tau^{-n}(\xi) = \phi_{\tau^{-1}(\xi)}^{-n}(x)$ and $\phi_{\tau^{-1}(\xi')}^{-n} \circ \gamma_{\xi'}^u(\xi') = \gamma_{\tau^{-n}(\xi)}^u \circ \tau^{-n}(\xi')$ we get

$$\begin{aligned}
\|\phi_{\tau^{-1}(\xi')}^{-n} \circ \gamma_{\xi'}^u(\xi') - \phi_{\tau^{-1}(\xi)}^{-n}(x)\| &= \\
&= \|\gamma_{\tau^{-n}(\xi)}^u \circ \tau^{-n}(\xi') - \gamma_{\tau^{-n}(\xi)}^u \circ \tau^{-n}(\xi)\| \leq \nu^{\alpha n} d_{\Sigma_k}(\xi', \xi)^{\alpha}.
\end{aligned}$$

This implies (2.5) goes to zero as n goes to infinity and therefore (ξ', x') belongs to $W^u((\xi, x); \Phi)$. The proof of Item (v) and of the proposition is now complete. \square

By construction, the sets of the partition $\mathcal{W}_{\Gamma_{\Phi}}^u$ are the *local strong unstable set* $W_{loc}^{uu}((\xi, x); \Phi)$ throughout the point (ξ, x) in Γ_{Φ} . We define the *(global) strong unstable set* of $(\xi, x) \in \Gamma_{\Phi}$ as

$$W^{uu}((\xi, x); \Phi) \stackrel{\text{def}}{=} \bigcup_{n \geq 0} \Phi^n(W_{loc}^{uu}(\Phi^{-n}(\xi, x); \Phi)).$$

Before stating the next proposition we denote by $\text{Per}(\Phi)$ the set of periodic points of Φ and that $\mathcal{P} : \Sigma_k \times M \rightarrow M$ is the projection on the fiber space.

Proposition 2.6. *Consider $\Phi = \tau \ltimes \phi_{\xi} \in \mathcal{S}$ with $\beta < 1$, $\alpha > 0$. Then*

- i) $W^{uu}((\xi, x); \Phi) = W^u((\xi, x); \Phi) \subset \Gamma_{\Phi}$ for all $(\xi, x) \in \Gamma_{\Phi}$, and
- ii) for all periodic point (ϑ, p) of Φ in $\Sigma_k \times \overline{D}$

$$K_{\Phi} \stackrel{\text{def}}{=} \overline{\mathcal{P}(\text{Per}(\Phi)) \cap D} = \overline{\mathcal{P}(W^u((\vartheta, p); \Phi))} = \mathcal{P}(\Gamma_{\Phi}) = g_{\Phi}(\Sigma_k).$$

Proof. To prove the inclusion $W^{uu}((\xi, x); \Phi) \subset W^u((\xi, x); \Phi)$ for all (ξ, x) in Γ_{Φ} , we take $(\xi', x') \in W^{uu}((\xi, x); \Phi)$ and show that

$$\lim_{n \rightarrow \infty} d(\Phi^{-n}(\xi', x'), \Phi^{-n}(\xi, x)) = 0. \tag{2.6}$$

Since $(\xi', x') \in W^{uu}((\xi, x); \Phi)$, by definition there are $m \in \mathbb{N}$ and $(\zeta, z) \in W_{loc}^{uu}(\Phi^{-m}(\xi, x); \Phi)$ such that $(\xi', x') = \Phi^m(\zeta, z)$. Let $(\eta, y) = \Phi^{-m}(\xi, x)$. Note that $(\eta, y) \in \Gamma_\Phi$, $(\zeta, z) \in W_{loc}^{uu}((\eta, y); \Phi)$, and

$$d(\Phi^{-n}(\xi', x'), \Phi^{-n}(\xi, x)) = d(\Phi^{-(n-m)}(\zeta, z), \Phi^{-(n-m)}(\eta, y)).$$

By Item (v) of Proposition 2.3, we have that $W_{loc}^{uu}((\eta, y); \Phi) \subset W^u((\eta, y); \Phi)$ and thus it follows (2.6).

Let us prove the converse inclusion: if $(\xi, x) \in \Gamma_\Phi$ then $W^u((\xi, x); \Phi) \subset W^{uu}((\xi, x); \Phi)$. Take $(\xi', x') \in W^u((\xi, x); \Phi)$ and we have to show that there is $m \in \mathbb{N}$ such that $\Phi^{-m}(\xi', x') \in W_{loc}^{uu}(\Phi^{-m}(\xi, x); \Phi)$. By definition of the unstable set,

$$\begin{aligned} \lim_{n \rightarrow \infty} d_{\Sigma_k}(\tau^{-n}(\xi'), \tau^{-n}(\xi)) &= 0 \quad \text{and} \\ \lim_{n \rightarrow \infty} \|\phi_{\tau^{-1}(\xi')}^{-n}(x') - \phi_{\tau^{-1}(\xi)}^{-n}(x)\| &= 0. \end{aligned} \quad (2.7)$$

Since $(\xi, x) \in \Gamma_\Phi$ then $\phi_{\tau^{-1}(\xi)}^{-n}(x) \in D$ for all $n \geq 0$. Thus there exists $m \in \mathbb{N}$ such that

$$\tau^{-m}(\xi') \in W_{loc}^u(\tau^{-m}(\xi); \tau) \quad \text{and} \quad \phi_{\tau^{-1}(\xi')}^{-(n+m)}(x') \in D, \quad \text{for every } n \geq m.$$

Write $(\eta, y) = \Phi^{-m}(\xi, x) \in \Gamma_\Phi$ and $(\eta', y') = \Phi^{-m}(\xi', x')$. Hence $y = g_\Phi(\eta)$ and $\phi_{\tau^{-n}(\eta)}^{n-i} \circ \phi_{\tau^{-1}(\eta)}^{-n}(y)$, $\phi_{\tau^{-n}(\eta')}^{n-i} \circ \phi_{\tau^{-1}(\eta')}^{-n}(y')$ belong to D for all $0 < i \leq n$. Recalling the definition of γ_η^u and $\gamma_{\eta'}^{u,n}$ we have that

$$\begin{aligned} \|y' - \gamma_\eta^u(\eta')\| &= \lim_{n \rightarrow \infty} \|\phi_{\tau^{-n}(\eta')}^n \circ \phi_{\tau^{-1}(\eta')}^{-n}(y') - \phi_{\tau^{-n}(\eta')}^n \circ \phi_{\tau^{-1}(\eta)}^{-n}(y)\| \\ &\leq \lim_{n \rightarrow \infty} \beta^n \|\phi_{\tau^{-1}(\eta')}^{-n}(y') - \phi_{\tau^{-1}(\eta)}^{-n}(y)\|. \end{aligned}$$

From (2.7) and since $\beta < 1$, we get this limit is zero and so $y' = \gamma_\eta^u(\eta')$. That is $\Phi^{-m}(\xi', x') \in W_{loc}^{uu}(\Phi^{-m}(\xi, x); \Phi)$ and therefore $(\xi', x') \in W^{uu}((\xi, x); \Phi)$, concluding our assertion.

Note, we proved that $W^{uu}((\xi, x); \Phi) = W^u(\xi, x; \Phi)$ for all $(\xi, x) \in \Gamma_\Phi$. Then by Proposition 2.3, $W^u(\xi, x; \Phi) \subset \Gamma_\Phi$ ending the first item of the proposition.

To prove Item (ii), consider a periodic point $\vartheta \in \Sigma_k$ of τ and note that $W^u(\vartheta; \tau)$ and $\text{Per}(\tau)$ are both dense in Σ_k . Using the conjugation in Proposition 2.2 and item (i), we get

$$\overline{\text{Per}(\Phi|_{\Gamma_\Phi})} = \Gamma_\Phi = \overline{W^u((\vartheta, g_\Phi(\vartheta)); \Phi)} = \overline{W^{uu}((\vartheta, g_\Phi(\vartheta)); \Phi)}. \quad (2.8)$$

If $(\vartheta, p) \in \Sigma_k \times \overline{D}$ is a periodic point of Φ , from the assumption $\phi_\xi(\overline{D}) \subset D$, we have that $\Phi^n(\vartheta, p) \in \Sigma_k \times D$ for all $n \in \mathbb{Z}$. Moreover, since g_Φ is the unique invariant graph of Φ restricted to $\Sigma_k \times \overline{D}$, by Item (ii) in Theorem 2.1 we get $p = g_\Phi(\vartheta)$. From this,

$$\text{Per}(\Phi|_{\Gamma_\Phi}) = \text{Per}(\Phi|_{\Sigma_k \times \overline{D}}) = \text{Per}(\Phi) \cap (\Sigma_k \times D). \quad (2.9)$$

Recalling that K_Φ is the closure of projection by \mathcal{P} of the periodic points of Φ in D and since the projection \mathcal{P} is a closed map and Σ_k is a compact set, Equations (2.8) and (2.9) imply that

$$\mathcal{P}(\Gamma_\Phi) = \overline{\mathcal{P}(W^u((\vartheta, p); \Phi))} = \overline{\mathcal{P}(\text{Per}(\Phi|_{\Gamma_\Phi}))} = \overline{\mathcal{P}(\text{Per}(\Phi))} \cap D \stackrel{\text{def}}{=} K_\Phi.$$

Finally, by definition $\mathcal{P}(\Gamma_\Phi) = g_\Phi(\Sigma_k)$ and from the above equation $K_\Phi = g_\Phi(\Sigma_k)$, completing the proof. \square

2.3. Continuation of the reference cube. In this subsection we will prove that K_Φ depends continuously with respect to Φ .

Given two subsets A and B of \overline{D} we define

$$d_H(A, B) \stackrel{\text{def}}{=} \sup\{d(a, B), d(b, A) : a \in A, b \in B\}, \quad A, B \in \mathcal{K}(\overline{D}).$$

We have the following properties of d_H .

- (H1) $d_H(\overline{A}, \overline{B}) = d_H(A, B)$,
- (H2) $d_H(T(A), T(B)) \leq \text{Lip}(T) d_H(A, B)$ where T is a Lipschitz map,
- (H3) if A_i and B_i are non-empty subsets for all i in a set of index I then

$$d_H\left(\bigcup_{i \in I} A_i, \bigcup_{i \in I} B_i\right) \leq \sup_{i \in I} d_H(A_i, B_i).$$

We consider the set $\mathcal{K}(\overline{D})$ whose elements are the compact subsets of \overline{D} endowed with the Hausdorff metric d_H obtaining a complete metric space. Define the map

$$\mathcal{L} : \mathcal{S} \rightarrow \mathcal{K}(\overline{D}), \quad \mathcal{L}(\Phi) \stackrel{\text{def}}{=} K_\Phi.$$

Note that by Proposition 2.6 we have $K_\Phi \in \mathcal{K}(\overline{D})$ for all $\Phi \in \mathcal{S}$ and thus this map is well defined.

Proposition 2.7. *Consider $\Phi = \tau \times \phi_\xi \in \mathcal{S}$ with $\beta < 1$, $\alpha > 0$. Then, for each $\varepsilon > 0$ there is $\delta > 0$ such that for every $\Psi = \tau \times \psi_\xi \in \mathcal{S}$ with*

$$d_{C^0}(\phi_\xi, \psi_\xi) < \delta \quad \text{it holds that} \quad d_H(K_\Phi, K_\Psi) = d_H(\mathcal{L}(\Phi), \mathcal{L}(\Psi)) < \varepsilon.$$

Proof. Fix a small $\varepsilon > \epsilon > 0$, let $\delta = \epsilon(1 - \beta)/2 > 0$. and take a fixed point $\vartheta \in \Sigma_k$ of τ . Since $\phi_\vartheta(\overline{D}) \subset D$, by Brouwer's fixed point theorem, there is $p_\Phi \in D$ such that $\phi_\vartheta(p_\Phi) = p_\Phi$. Since the map ϕ_ϑ is a contraction, this fixed point is hyperbolic. Thus (ϑ, p_Φ) is a fixed point of Φ in $\Sigma_k \times D$.

If δ is small enough then for every $\Psi = \tau \times \psi_\xi$ with $d_{C^0}(\phi_\xi, \psi_\xi) < \delta$ there is $p_\Psi \in D$ close to p_Φ which is a fixed point of ψ_ϑ . Then $(\vartheta, p_\Psi) \in \Gamma_\Psi$ is a fixed point of Ψ , called the continuation of $(\vartheta, p_\Phi) \in \Gamma_\Phi$ for Ψ . Take $\Theta = \tau \times \theta_\xi \in \{\Phi, \Psi\}$, by Proposition 2.6 the strong unstable set and the unstable set of (ϑ, p_Θ) are equal. Thus

$$\mathcal{P}(W^u((\vartheta, p_\Theta); \Theta)) = \bigcup_{n \geq 0} \mathcal{P} \circ \Theta^n(W_{loc}^{uu}((\vartheta, p_\Theta); \Theta)).$$

By Item (i) of Proposition 2.3, the graph set of $\gamma_{\vartheta, p_\Theta}^u : W_{loc}^u(\vartheta; \tau) \rightarrow M$ is the local strong unstable set of (ϑ, p_Θ) for Θ . Then for each $n \geq 0$,

$$\mathcal{P}(\Theta^n(W_{loc}^{uu}((\vartheta, p_\Theta); \Theta))) = \{\theta_\xi^n \circ \gamma_{\vartheta, p_\Theta}^u(\xi) : \xi \in W_{loc}^u(\vartheta; \tau)\} \stackrel{\text{def}}{=} E_n(\Theta).$$

Hence by Proposition 2.6

$$K_\Theta = \overline{\mathcal{P}(W^u((\vartheta, p_\Theta); \Theta))} = \overline{\bigcup_{n \geq 0} E_n(\Theta)}, \quad \Theta = \Phi, \Psi.$$

By Item (H3) of properties of d_H

$$d_H(K_\Phi, K_\Psi) \leq \sup_{n \geq 0} d_H(E_n(\Phi), E_n(\Psi)).$$

On the other hand, for each $n \geq 0$, we have that

$$d_H(E_n(\Phi), E_n(\Psi)) \leq \sup_{\xi \in W_{loc}^u(\vartheta; \tau)} \|\phi_\xi^n \circ \gamma_{\vartheta, p_\Phi}^u(\xi) - \psi_\xi^n \circ \gamma_{\vartheta, p_\Psi}^u(\xi)\|. \quad (2.10)$$

Fix $\xi \in W_{loc}^u(\vartheta; \tau)$, first we will estimate (2.10) for $n = 0$.

Claim 2.8. $\gamma_\vartheta^u(\xi) = \phi_{\tau^{-m}(\xi)}^m \circ \gamma_\vartheta^u \circ \tau^{-m}(\xi)$ for all $m \in \mathbb{N}$ and $\xi \in W^u(\vartheta; \tau)$.

Proof. Recall that $(\phi_{\tau^{-1}(\xi)}^m)^{-1} = \phi_{\tau^{-1}(\xi)}^{-m}$, by the notation in (1.4) and Item (iv) in Proposition 2.3 we get:

$$\begin{aligned} \phi_{\tau^{-m}(\xi)}^{-m} \circ \gamma_\vartheta^u(\xi) &= \phi_{\tau^{-m}(\xi)}^{-1} \circ \cdots \circ \underbrace{\phi_{\tau^{-1}(\xi)}^{-1} \circ \gamma_\vartheta^u(\xi)}_{\phi_{\tau^{-1}(\xi)}^{-1} \circ \gamma_{\tau^{-1}(\vartheta)}^u \circ \tau^{-1}(\xi)} \\ &= \phi_{\tau^{-m}(\xi)}^{-1} \circ \cdots \circ \underbrace{\phi_{\tau^{-2}(\xi)}^{-1} \circ \gamma_{\tau^{-1}(\vartheta)}^u \circ \tau^{-1}(\xi)}_{\phi_{\tau^{-2}(\xi)}^{-1} \circ \gamma_{\tau^{-2}(\vartheta)}^u \circ \tau^{-2}(\xi)} \\ &= \phi_{\tau^{-m}(\xi)}^{-1} \circ \cdots \circ \phi_{\tau^{-3}(\xi)}^{-1} \circ \gamma_{\tau^{-2}(\vartheta)}^u \circ \tau^{-2}(\xi) \\ &\vdots \\ &= \gamma_{\tau^{-m}(\vartheta)}^u \circ \tau^{-m}(\xi) = \gamma_\vartheta^u \circ \tau^{-m}(\xi). \quad \square \end{aligned}$$

Using the claim above, the triangle inequality and the β -contraction of ϕ_ξ ,

$$\begin{aligned} \|\gamma_{\vartheta, p_\Phi}^u(\xi) - \gamma_{\vartheta, p_\Psi}^u(\xi)\| &= \\ &= \|\phi_{\tau^{-m}(\xi)}^m \circ \gamma_{\vartheta, p_\Phi}^u \circ \tau^{-m}(\xi) - \psi_{\tau^{-m}(\xi)}^m \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi)\| \\ &\leq \|\phi_{\tau^{-m}(\xi)}^m \circ \gamma_{\vartheta, p_\Phi}^u \circ \tau^{-m}(\xi) - \phi_{\tau^{-m}(\xi)}^m \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi)\| + \\ &\quad + \|\phi_{\tau^{-m}(\xi)}^m \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi) - \psi_{\tau^{-m}(\xi)}^m \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi)\| \\ &\leq \beta^m \|\gamma_{\vartheta, p_\Phi}^u \circ \tau^{-m}(\xi) - \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi)\| + \\ &\quad + \|\phi_{\tau^{-m}(\xi)}^m \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi) - \psi_{\tau^{-m}(\xi)}^m \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi)\|. \end{aligned}$$

We may write the last inequality in the form $\beta^m(A) + (B)$. By continuity and since ξ is in the local unstable manifold of the fixed point ϑ for τ we get

$$(A) = \|\gamma_{\vartheta, p_\Phi}^u \circ \tau^{-m}(\xi) - \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi)\| \xrightarrow{m \rightarrow \infty} \|p_\Phi - p_\Psi\|.$$

Hence the first term in the sum goes to zero as $m \rightarrow \infty$. To estimate the term (B) recall that $d_{C^0}(\phi_\xi, \psi_\xi) < \delta$. Thus

$$\begin{aligned} \|\phi_{\tau^{-1}(\xi)} \circ \phi_{\tau^{-m}(\xi)}^{m-1} \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi) - \psi_{\tau^{-1}(\xi)} \circ \psi_{\tau^{-m}(\xi)}^{m-1} \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi)\| &\leq \\ &\leq \|\phi_{\tau^{-1}(\xi)} \circ \phi_{\tau^{-m}(\xi)}^{m-1} \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi) - \psi_{\tau^{-1}(\xi)} \circ \phi_{\tau^{-m}(\xi)}^{m-1} \circ \gamma_{\vartheta, p_\Phi}^u \circ \tau^{-m}(\xi)\| \\ &\quad + \|\psi_{\tau^{-1}(\xi)} \circ \phi_{\tau^{-m}(\xi)}^{m-1} \circ \gamma_{\vartheta, p_\Phi}^u \circ \tau^{-m}(\xi) - \psi_{\tau^{-1}(\xi)} \circ \psi_{\tau^{-m}(\xi)}^{m-1} \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi)\| \\ &\leq \delta + \beta \|\phi_{\tau^{-m}(\xi)}^{m-1} \circ \gamma_{\vartheta, p_\Phi}^u \circ \tau^{-m}(\xi) - \psi_{\tau^{-m}(\xi)}^{m-1} \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi)\|. \end{aligned}$$

Arguing inductively we obtain

$$\|\phi_{\tau^{-m}(\xi)}^m \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi) - \psi_{\tau^{-m}(\xi)}^m \circ \gamma_{\vartheta, p_\Psi}^u \circ \tau^{-m}(\xi)\| \leq \delta \sum_{k=0}^{m-1} \beta^k \leq \frac{\delta}{1-\beta}.$$

Putting together the estimates for (A) and (B) we get $\|\gamma_{\vartheta, p_\Phi}^u(\xi) - \gamma_{\vartheta, p_\Psi}^u(\xi)\| \leq \delta(1-\beta)^{-1} < \epsilon$, proving (2.10) for $n = 0$.

Now, for each $n \geq 1$, with a similar calculation we obtain that

$$\begin{aligned} \|\phi_\xi^n \circ \gamma_{\vartheta, p_\Phi}^u(\xi) - \psi_\xi^n \circ \gamma_{\vartheta, p_\Psi}^u(\xi)\| &\leq \\ &\leq \beta^n \|\gamma_{\vartheta, p_\Phi}^u(\xi) - \gamma_{\vartheta, p_\Psi}^u(\xi)\| + \|\phi_\xi^n \circ \gamma_{\vartheta, p_\Psi}^u(\xi) - \psi_\xi^n \circ \gamma_{\vartheta, p_\Psi}^u(\xi)\|. \end{aligned}$$

Arguing analogously (using the triangle inequality and $d_{C^0}(\phi_\xi, \psi_\xi) < \delta$), we get that $\|\phi_\xi^n \circ \gamma_{\vartheta, p_\Psi}^u(\xi) - \psi_\xi^n \circ \gamma_{\vartheta, p_\Psi}^u(\xi)\| \leq \delta(1 - \beta)^{-1}$. Putting together these estimates and using that $\beta^n < 1$, we have that

$$\|\phi_\xi^n \circ \gamma_{\vartheta, p_\Phi}^u(\xi) - \psi_\xi^n \circ \gamma_{\vartheta, p_\Psi}^u(\xi)\| \leq \beta^n \frac{\delta}{1 - \beta} + \frac{\delta}{1 - \beta} \leq \frac{2\delta}{1 - \beta} = \epsilon.$$

By (2.10) this implies $d_H(K_\Phi, K_\Psi) \leq \sup_{n \in \mathbb{N}} d_H(E_n(\Phi), E_n(\Psi)) \leq \epsilon < \varepsilon$, ending the proof of the proposition. \square

The following theorem summarizes the results obtaining in this section:

Theorem 2.9. *Consider $\Phi \in \mathcal{S}_{k, \lambda, \beta}^{0, \alpha}(D)$ with $\beta < 1$ and $\alpha > 0$. Then*

$$\Gamma_\Phi \stackrel{\text{def}}{=} \bigcap_{n \in \mathbb{Z}} \Phi^n(\Sigma_k \times \overline{D}) = \bigcap_{n \in \mathbb{N}} \Phi^n(\Sigma_k \times \overline{D}).$$

Moreover, the following holds:

- i) the restriction of Φ to Γ_Φ is conjugate to the full shift τ of k symbols,
- ii) $W^{uu}((\xi, x); \Phi) = W^u((\xi, x); \Phi) \subset \Gamma_\Phi$ for all $(\xi, x) \in \Gamma_\Phi$,
- iii) there exists a unique continuous function $g_\Phi : \Sigma_k \rightarrow \overline{D}$ such that for every periodic point (ϑ, p) of Φ it holds,

$$\mathcal{P}(\Gamma_\Phi) = \overline{\text{Per}(\Phi) \cap \overline{D}} \stackrel{\text{def}}{=} K_\Phi \in \mathcal{K}(D)$$

and

$$\Gamma_\Phi = \overline{W^u((\vartheta, p); \Phi)} = \{(\xi, g_\Phi(\xi)) : \xi \in \Sigma_k\},$$

- iv) the map $\mathcal{L} : \mathcal{S}_{k, \lambda, \beta}^{0, \alpha}(D) \rightarrow \mathcal{K}(\overline{D})$ given by $\mathcal{L}(\Phi) = K_\Phi$ is continuous.

As of consequence of the above theorem, one has that the *local unstable set* of Γ_Φ for Φ defined as

$$W_{loc}^{uu}(\Gamma_\Phi; \Phi) \stackrel{\text{def}}{=} \bigcup_{(\xi, x) \in \Gamma_\Phi} W_{loc}^{uu}((\xi, x); \Phi).$$

is precisely Γ_Φ . Indeed,

$$\Gamma_\Phi \subset W_{loc}^{uu}(\Gamma_\Phi; \Phi) \subset \bigcup_{(\xi, x) \in \Gamma_\Phi} W^u((\xi, x); \Phi) \subset \Gamma_\Phi.$$

Moreover,

$$W^u(\Gamma_\Phi; \Phi) \stackrel{\text{def}}{=} \{(\xi, x) \in \Sigma_k \times M : \lim_{n \rightarrow \infty} d(\Phi^{-n}(\xi, x), \Gamma_\Phi) = 0\} = \Gamma_\Phi.$$

3. SYMBOLIC BLENDERS IN THE ONE-STEP SETTING

Let us introduce the definition given in [NP12] for one-step symbolic blenders. First consider the subset $\mathcal{Q} = \mathcal{Q}_{k,\lambda,\beta}^0(D)$ of $\mathcal{S} = \mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ consisting of one-step skew-product maps.

Definition 3.1 (One-step symbolic *cs*-blender). *Consider a one-step skew-product map $\Phi \in \mathcal{Q}$ with $\beta < 1$.*

*The maximal invariant set Γ_Φ of Φ in $\Sigma_k \times \overline{D}$ is a one-step symbolic *cs*-blender if there are a non-empty open set $B \subset D$, a fixed point $(\vartheta, p) \in \Sigma_k \times D$ of Φ and a neighborhood \mathcal{V} of Φ in \mathcal{Q} such that for every $\Psi \in \mathcal{V}$ it holds*

$$W^u((\vartheta, p_\Psi); \Psi) \cap (W_{loc}^s(\xi; \tau) \times U) \neq \emptyset, \quad (3.1)$$

for every $\xi \in \Sigma_k$ and every non-empty open subset U in B . Here (ϑ, p_Ψ) is the continuation of the fixed point (ϑ, p) for Ψ .

Observe that Definition 1.3 implies Definition 3.1. Let (ϑ, p) be a fixed point of $\Phi \in \mathcal{Q}$. Note that for $(\xi, x) \in \Sigma_k \times B$ and for any $\delta > 0$, the set $W_{loc}^s(\xi; \tau) \times \{x\}$ is a δ -almost horizontal disk. Then by Theorem 2.9 and Definition 1.3, the closure of $W^u((\vartheta, p_\Psi); \Psi)$ meets $W_{loc}^s(\xi; \tau) \times \{x\}$ for every perturbation Ψ of Φ in \mathcal{Q} . Hence the set $W^u((\vartheta, p_\Psi); \Psi)$ meets $W_{loc}^s(\xi; \tau) \times U$, for every open set U with $x \in U$.

In this section, we prove the existence of symbolic blenders in the one-step setting. We begin studying the relation between one-step skew-products and their associated IFS. To construct symbolic one-step blenders we use the covering property and the Hutchinson attractor of the associated IFS.

3.1. One-step skew-products and IFS's. Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ and let $\text{IFS}(\Phi)$ be the associated iterated function system. The orbit of a point $x \in M$ for $\text{IFS}(\Phi)$ is the set

$$\text{Orb}_\Phi(x) \stackrel{\text{def}}{=} \{\phi(x) : \phi \in \text{IFS}(\phi_1, \dots, \phi_k)\}.$$

Next proposition shows that if (ϑ, p) is a fixed point of Φ then $\text{Orb}_\Phi(p)$ is the projection onto the fiber space of the strong unstable set of (ϑ, p) . This result was proved in [NP12, Proposition 2.16], since the proof is short for completeness we include it here. A consequence of this proposition is that the density property (3.1) of the strong unstable set in Definition 3.1 of one-step symbolic blenders is reduced to the density of the orbit of the “fixed point” p for the associated iteration function system.

Proposition 3.2. *Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ and let (ϑ, p) be a fixed point of Φ . Then $\mathcal{P}(W^{uu}(\vartheta, p); \Phi) = \text{Orb}_\Phi(p)$.*

Proof. Since (ϑ, p) is a fixed point of Φ then

$$W^{uu}(\vartheta, p); \Phi = \bigcup_{n=0}^{\infty} \Phi^n(W_{loc}^{uu}((\vartheta, p); \Phi)).$$

On the other hand, we have for each $n \geq 1$

$$\Phi^n(W_{loc}^{uu}((\vartheta, p); \Phi) = \{(\tau^n(\zeta), \phi_{\tau^{n-1}(\zeta)} \circ \dots \circ \phi_\zeta(p)) : \zeta \in W_{loc}^u(\vartheta; \tau)\}.$$

Since Φ is one-step, $\phi_{\tau^i(\zeta)} = \phi_{\zeta_i}$ for all $i \geq 0$. Note that $\phi_\zeta(p) = \phi_\vartheta(p) = p$ and then

$$\begin{aligned} \mathcal{P}(\Phi^n(W_{loc}^{uu}((\vartheta, p); \Phi))) &= \{\phi_{\tau^{n-1}(\zeta)} \circ \cdots \circ \phi_{\tau(\zeta)}(p) : \zeta \in W_{loc}^u(\vartheta; \tau)\} \\ &= \{\phi_{i_{n-1}} \circ \cdots \circ \phi_{i_1}(p) : i_j \in \{1, \dots, k\}, 1 \leq j < n\}. \end{aligned}$$

Hence the projection on the fiber space of the strong unstable set is $\text{Orb}_\Phi(p)$, concluding the proof of the proposition. \square

A neighborhood \mathcal{V} of a one-step map Φ in \mathcal{Q} is a neighborhood in the topology of \mathcal{S} intersected with \mathcal{Q} . Since the topology of \mathcal{S} is induced by the distance in (1.3) and noting that for every $\Psi \in \mathcal{Q}$ its Hölder constant is $C_\Psi = 0$, it follows that $\Psi = \tau \times (\psi_1, \dots, \psi_k)$ and $\Phi = \tau \times (\phi_1, \dots, \phi_k)$ are δ -close if

$$d_{\mathcal{Q}}(\Psi, \Phi) = \max_{i=1, \dots, k} d_{C^0}(\psi_i|_D, \phi_i|_D) < \delta.$$

A periodic point (ϑ, p) of a skew-product map $\Phi = \tau \times \phi_\xi$ is *fiber-hyperbolic* for Φ if p is a hyperbolic fixed point of ϕ_ξ^n , where n is the period of (ϑ, p) . We analogously define *fiber-attractors* and *fiber-repellors*.

Proposition 3.3. *Consider $\Phi \in \mathcal{Q}$, a non-empty open set $B \subset D$, and a fiber-hyperbolic fixed point $(\vartheta, p) \in \Sigma_k \times D$ of Φ . The following properties are equivalent:*

i) *there is a neighborhood \mathcal{V} of Φ in \mathcal{Q} such that for every $\Psi \in \mathcal{V}$,*

$$W^{uu}((\vartheta, p_\Psi); \Psi) \cap (W_{loc}^s(\xi; \tau) \times U) \neq \emptyset,$$

for every $\xi \in \Sigma_k$ and non-empty open set $U \subset B$. Here (ϑ, p_Ψ) is the continuation of (ϑ, p) for Ψ .

ii) *$B \subset \overline{\text{Orb}_\Psi(p_\Psi)}$ for every $\Psi \in \mathcal{Q}$ close to Φ .*

Proof. From Proposition 3.2, for a fixed point (ϑ, p_Ψ) of Ψ , we have that $\mathcal{P}(W^{uu}((\vartheta, p_\Psi); \Psi)) = \text{Orb}_\Psi(p_\Psi)$. Therefore, Item (i) implies Item (ii).

For the converse, take the fixed point (ϑ, p_Ψ) of $\Psi = \tau \times (\psi_1, \dots, \psi_k)$ close to $\Phi = \tau \times (\phi_1, \dots, \phi_k)$ and fix $U \subset B$ and $\xi \in \Sigma_k$. By Item (ii), there is $\psi_{i_n} \circ \cdots \circ \psi_{i_1} \in \text{IFS}(\Psi)$ such that the point $x = \psi_{i_n} \circ \cdots \circ \psi_{i_1}(p_\Psi) \in U$. Take $\zeta = (\dots \vartheta_{-1} \vartheta_0, i_1, \dots, i_n; \xi_0, \xi_1, \dots)$, and note that $(\zeta, x) \in W_{loc}^s(\xi; \tau) \times U$. It is enough to see that $(\zeta, x) \in W^{uu}((\vartheta, p_\Psi); \Phi)$. Since (ϑ, p_Ψ) is a fixed point of Ψ , by the choice of x we have that

$$\Psi^{-n-1}(\zeta, x) = ((\dots, \vartheta_{-1}; \vartheta_0, i_1, \dots, i_n, \xi_0, \xi_1, \dots), p_\Psi) \in W_{loc}^u(\vartheta; \tau) \times \{p_\Psi\}.$$

Since $W_{loc}^u(\vartheta; \tau) \times \{p_\Psi\} = W_{loc}^{uu}((\vartheta, p_\Psi); \Psi)$ then

$$(\zeta, x) \in \Psi^{n+1}(W_{loc}^{uu}((\vartheta, p_\Psi); \Psi)) \subset W^{uu}((\vartheta, p_\Psi); \Psi).$$

Hence

$$(\zeta, x) \in W^{uu}((\vartheta, p_\Psi); \Psi) \cap (W_{loc}^s(\xi; \tau) \times U),$$

completing the proof of the proposition. \square

Remark 3.4. If (ϑ, p) in Proposition 3.3 is a fiber-attracting fixed point of $\Phi = \tau \times (\phi_1, \dots, \phi_k)$ with B contained in the attracting region of p for $\text{IFS}(\Phi)$, then Item (ii) is equivalent to

$$B \subset \overline{\text{Orb}_\Psi(x)}, \quad \text{for every } x \in B \text{ and every } \Psi \in \mathcal{Q} \text{ close to } \Phi. \quad (3.2)$$

To see why this remark is true first note that Equation (3.2) immediately implies Item (ii) (just take $x = p$). For the converse take a perturbation $\Psi = \tau \times (\psi_1, \dots, \psi_k)$ of Φ in \mathcal{Q} , a non-empty open set U in B , and $x \in B$. By hypotheses, there is $\psi \in \text{IFS}(\Psi)$ such that $\psi(p_\Psi) \in U$. As U is open there is a neighborhood V of p_Ψ such that $\psi(V) \subset U$. If Ψ is close enough to Φ then B is also in the attracting region of p_Ψ for $\psi_{\vartheta} = \psi_i$ where $\vartheta_0 = i$. Thus there is $n \in \mathbb{N}$ such that $\psi_i^n(x) \in V$ and hence $\psi \circ \psi_i^n(x) \in U$, proving (3.2).

Motivated by (3.2), we give the following definition:

Definition 3.5 (Blending regions). *Let $\Phi = \tau \times (\phi_1, \dots, \phi_k) \in \mathcal{Q}$. A non-empty open set $B \subset M$ is called a blending region for Φ (or for the IFS(Φ)) if for every $\Psi = \tau \times (\psi_1, \dots, \psi_k)$ close to Φ*

$$B \subset \overline{\text{Orb}_\Psi(x)} \quad \text{for all } x \in B.$$

Proposition 3.6. *Let $\Phi = \tau \times (\phi_1, \dots, \phi_k) \in \mathcal{Q}$ and consider a blending region $B \subset D$ of Φ . Suppose that there are a hyperbolic fixed point $p \in D$ of some ϕ_i and a map $\phi \in \text{IFS}(\Phi)$ with $\phi(p) \in B$. Then the maximal invariant set of Φ in $\Sigma_k \times \overline{D}$ is a one-step symbolic blender.*

Proof. By Proposition 3.3, it is enough to see that $B \subset \overline{\text{Orb}_\Psi(p_\Psi)}$, for every $\Psi = \tau \times (\psi_1, \dots, \psi_k)$ close to Φ , where p_Ψ the continuation of p for Ψ . By hypothesis, there are i_n, \dots, i_1 such that $\phi_{i_n} \circ \dots \circ \phi_{i_1}(p) \in B$. Since B is an open set, if $\Psi = \tau \times (\psi_1, \dots, \psi_k)$ is close enough to Φ then $\psi_{i_n} \circ \dots \circ \psi_{i_1}(p_\Psi) \in B$. As B is a blending region for IFS(Φ) it follows that

$$B \subset \overline{\text{Orb}_\Psi(\psi_{i_n} \circ \dots \circ \psi_{i_1}(p_\Psi))} \subset \overline{\text{Orb}_\Psi(p_\Psi)}.$$

This concludes the proof of the proposition. \square

3.2. Blending regions for contracting IFS: Hutchinson attractor. Given an one-step map $\Phi = \tau \times (\phi_1, \dots, \phi_k)$, we define $\text{Per}(\text{IFS}(\Phi))$ as the projection of $\mathcal{P}(\text{Per}(\Phi))$ in the fiber space, that is the set of fixed points of the maps in $\text{IFS}(\Phi)$. Associated with Φ or with $\text{IFS}(\Phi)$, the *Hutchinson's operator* is defined by

$$\mathcal{G}_\Phi: \mathcal{K}(\overline{D}) \rightarrow \mathcal{K}(\overline{D}), \quad \mathcal{G}_\Phi(A) \stackrel{\text{def}}{=} \phi_1(A) \cup \dots \cup \phi_k(A),$$

where $\mathcal{K}(\overline{D})$ denotes the set of compact subsets of \overline{D} and $A \in \mathcal{K}(\overline{D})$. If the maps ϕ_i are contractions, then the map \mathcal{G}_Φ is also a contraction. This fact leads to the following result:

Proposition 3.7 ([Wil71, Hut81]). *Let $\Phi \in \mathcal{Q}$ with $\beta < 1$. Then there exists a unique compact set $K_{\mathcal{G}_\Phi} \in \mathcal{K}(\overline{D})$ such that*

$$K_{\mathcal{G}_\Phi} = \mathcal{G}_\Phi(K_{\mathcal{G}_\Phi}) = \overline{\text{Per}(\text{IFS}(\Phi)) \cap \overline{D}} = K_\Phi.$$

Moreover, the set $K_{\mathcal{G}_\Phi}$ depends continuously (in the set \mathcal{Q}) on the map Φ and is the global attractor of \mathcal{G}_Φ , that is, for every $A \in \mathcal{K}(\overline{D})$ it holds

$$\lim_{m \rightarrow \infty} d_H(\mathcal{G}_\Phi^m(A), K_{\mathcal{G}_\Phi}) = 0.$$

We call the compact set $K_{\mathcal{G}_\Phi}$ (in what follows denoted by K_Φ) the *Hutchinson attractor* of the contracting one-step map Φ or of its associated $\text{IFS}(\Phi)$.

Let us recall that given $x \in D$, its orbit for $\text{IFS}(\Phi)$ is defined by

$$\text{Orb}_\Phi(x) = \{\phi(x) : \phi \in \text{IFS}(\Phi)\} = \{\mathcal{G}_\Phi^n(x) : n \geq 0\}.$$

By Proposition 3.7, $\mathcal{G}_\Phi^m(x) \xrightarrow{m \rightarrow \infty} K_\Phi$ for all $x \in D$ and thus

$$K_\Phi \subset \overline{\text{Orb}_\Phi(x)}. \quad (3.3)$$

We now have the following consequences of Proposition 3.7:

Corollary 3.8. *Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}$ with $\beta < 1$ and let K_Φ be the Hutchinson attractor.*

i) *For every $A \in \mathcal{K}(\overline{D})$ with $A \subset \mathcal{G}_\Phi(A)$ it holds that*

$$A \subset K_\Phi \subset \overline{\text{Orb}_\Phi(x)} \quad \text{for all } x \in \overline{D}.$$

ii) *For every $p \in K_\Phi$ there is a sequence $(\sigma_n)_{n \in \mathbb{N}} \in \{1, \dots, k\}^\mathbb{N}$ such that*

$$\phi_{\sigma_n}^{-1} \circ \dots \circ \phi_{\sigma_1}^{-1}(p) \in K_\Phi \quad \text{for all } n \in \mathbb{N}.$$

iii) *For each open set V such that $V \cap K_\Phi \neq \emptyset$ there exist $n \in \mathbb{N}$ and $(i_1, \dots, i_n) \in \{1, \dots, k\}^n$ such that $\phi_{i_n} \circ \dots \circ \phi_{i_1}(K_\Phi) \subset V$.*

Proof. To prove the first item note that by hypothesis, $A \subset \mathcal{G}_\Phi(A) \subset \dots \subset \mathcal{G}_\Phi^m(A)$ for all $m \geq 1$. Since $\mathcal{G}_\Phi^m(A) \rightarrow K_\Phi$ this implies that $A \subset K_\Phi$. By (3.3), one has that $K_\Phi \subset \overline{\text{Orb}_\Phi(x)}$ for all $x \in \overline{D}$ proving (i).

To prove Item (ii) recall that by Proposition 3.7, $K_\Phi = \phi_1(K_\Phi) \cup \dots \cup \phi_k(K_\Phi)$. Thus given any $p \in K_\Phi$ there exists $\sigma_1 \in \{1, \dots, k\}$ such that $\phi_{\sigma_1}^{-1}(p) \in K_\Phi$. Arguing inductively, we get a sequence $(\sigma_n)_{n \in \mathbb{N}}$ with $\phi_{\sigma_n}^{-1} \circ \dots \circ \phi_{\sigma_1}^{-1}(p) \in K_\Phi$ for all $n \in \mathbb{N}$.

To prove Item (iii), fix an open set V with $V \cap K_\Phi \neq \emptyset$. Consider $i \in \{1, \dots, k\}$ and the fixed point s of ϕ_i . By the first item we have $K_\Phi \subset \overline{\text{Orb}_\Phi(s)}$. Hence, there are $m \in \mathbb{N}$ and $(\sigma_1, \dots, \sigma_m) \in \{1, \dots, k\}^m$ such that $\phi_{\sigma_m} \circ \dots \circ \phi_{\sigma_1}(s) \in V$ and thus $\phi_{\sigma_1}^{-1} \circ \dots \circ \phi_{\sigma_m}^{-1}(V)$ is a neighborhood of s . Since ϕ_i^{-1} is an expansion, the set $\phi_{\sigma_1}^{-1} \circ \dots \circ \phi_{\sigma_m}^{-1}(V)$ contains the repelling point s of ϕ_i^{-1} , and as K_Φ is bounded, there exists $\ell \in \mathbb{N}$ such that $K_\Phi \subset \phi_i^{-\ell} \circ \phi_{\sigma_1}^{-1} \circ \dots \circ \phi_{\sigma_m}^{-1}(V)$. Now it is enough to take $n = \ell + m$ and the sequence $(i, \dots, i, \sigma_1, \dots, \sigma_m)$. This completes the proof of the corollary. \square

The next result shows that an open set satisfying the covering property for a contractive IFS is a blending region.

Corollary 3.9. *Consider $\Phi \in \mathcal{Q}$ with $\beta < 1$. Let $B \subset \overline{D}$ be a non-empty bounded open set satisfying the covering property for $\text{IFS}(\Phi)$. Then for every $\Psi \in \mathcal{Q}$ close enough to Φ one has that $\overline{B} \subset K_\Psi \subset \overline{\text{Orb}_\Psi(x)}$, for all $x \in \overline{D}$, where K_Ψ is the Hutchinson attractor of Ψ .*

The corollary above and Proposition 3.6 imply that the covering property generates one-step symbolic blenders (Definition 3.1).

Corollary 3.10. *Let $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}$ with $\beta < 1$ and let $B \subset \overline{D}$ be a non-empty open set satisfying the covering property:*

$$\overline{B} \subset \phi_1(B) \cup \dots \cup \phi_k(B).$$

Then the maximal invariant set of Φ is a one-step symbolic blender.

Proof of Corollary 3.9. If $\Psi \in \mathcal{Q}$ is close to Φ then $d_H(\mathcal{G}_\Psi(\overline{B}), \mathcal{G}_\Phi(\overline{B}))$ is small. From this proximity and since $\mathcal{G}_\Phi(B)$ is open, one has that $\overline{B} \subset \mathcal{G}_\Psi(B) \subset \mathcal{G}_\Psi(\overline{B})$. Inductively we get

$$\overline{B} \subset \mathcal{G}_\Psi^m(\overline{B}), \quad \text{for all } m \geq 0.$$

Since the Hutchinson attractor K_Ψ of Ψ is closed and $d_H(\mathcal{G}_\Psi^m(\overline{B}), K_\Psi) \rightarrow 0$, then $\overline{B} \subset K_\Psi \subset \overline{\text{Orb}_\Psi(x)}$ for all $x \in \overline{B}$. \square

4. SYMBOLIC BLENDERS IN THE HÖLDER SETTING

In this section we will prove the following result:

Theorem 4.1. *Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ with $\alpha > 0$ and $\nu^\alpha < \lambda < 1$ and let $B \subset D$ be an open set. Then, B satisfies the covering property for $\text{IFS}(\Phi)$ if and only if there are $\delta > 0$ and a neighborhood \mathcal{V} of Φ in $\mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ such that for every $\Psi \in \mathcal{V}$ it holds*

$$\Gamma_\Psi^+(B) \cap H^s \neq \emptyset \quad \text{for every } \delta\text{-horizontal disk } H^s \text{ in } \Sigma_k \times B, \quad (4.1)$$

where $\Gamma_\Psi^+(B)$ is the forward maximal invariant set of Ψ in $\Sigma_k \times B$.

The above theorem proves Theorem A. Indeed, if $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$, with $\beta < 1$, then $\phi_i(\overline{D}) \subset D$ for all $i = 1, \dots, k$. Hence, if $\Psi = \tau \ltimes \psi_\xi$ is close enough to Φ then $\psi_\xi(\overline{D}) \subset D$ for all $\xi \in \Sigma_k$ and thus

$$\Gamma_\Psi^+(B) \stackrel{\text{def}}{=} \bigcap_{n \geq 0} \Psi^n(\Sigma_k \times B) \subset \bigcap_{n \in \mathbb{Z}} \Psi^n(\Sigma_k \times \overline{D}) \stackrel{\text{def}}{=} \Gamma_\Psi. \quad (4.2)$$

Since by Theorem 2.9 one has that $W_{loc}^{uu}(\Gamma_\Psi; \Psi) = \Gamma_\Psi$, Theorem 4.1 and Equation 4.2 imply the existence of symbolic blender (Theorem A).

In order to prove Theorem 4.1, firstly we introduce some notation and preliminary results.

4.1. Main lemma: Given a finite word $\bar{\omega} = \omega_{-m} \dots \omega_{-1} \omega_0 \omega_1 \dots \omega_n$, where $m, n \geq 0$ and $\omega_i \in \{1, \dots, k\}$, we define the *bi-lateral cylinder* by

$$\mathcal{C}_{\bar{\omega}} \stackrel{\text{def}}{=} \{\xi \in \Sigma_k : \xi_j = \omega_j, \quad -m \leq j \leq n\}.$$

Given $\zeta \in \Sigma_k$ and a finite word $\bar{\omega} := \bar{\omega}_{-n} = \omega_{-n} \dots \omega_{-1}$, where $n \geq 1$ and $\omega_i \in \{1, \dots, k\}$, we define the *relative cylinder* by

$$\mathcal{C}_{\bar{\omega}}(\zeta) \stackrel{\text{def}}{=} \{\xi \in W_{loc}^s(\zeta; \tau) : \xi_{-i} = \omega_{-i}, \text{ for } i = 1, \dots, n\}. \quad (4.3)$$

Recall that $\mathcal{S} = \mathcal{S}_{k,\lambda,\beta}^{0,\alpha}(D)$ is the set of symbolic skew-product maps in Definition 1.5. In what follows $\nu^\alpha < \lambda < 1$, $\alpha > 0$ and there is no restriction on β . In the next lemma we estimate the distance between the backward orbits of a point x when iterated by different maps ψ_ξ^{-1} .

Lemma 4.2. *Consider $\Psi = \tau \ltimes \psi_\xi \in \mathcal{S}$ with $\alpha > 0$, $\nu^\alpha < \lambda < 1$ a word $\bar{\omega} = \omega_{-n} \dots \omega_0 \dots \omega_n$, and a point $x \in \overline{D}$ such that for every $\zeta \in \mathcal{C}_{\bar{\omega}}$ one has that $\psi_{\tau^{-1}(\zeta)}^{-j}(x) \in \overline{D}$ for every $1 \leq j \leq n$. Then*

$$\|\psi_{\tau^{-1}(\xi)}^{-i}(x) - \psi_{\tau^{-1}(\zeta)}^{-i}(x)\| < C_\Psi \nu^{-\alpha i} \sum_{j=0}^{i-1} (\lambda^{-1} \nu^\alpha)^j d_{\Sigma_k}(\xi, \zeta)^\alpha$$

for all $1 \leq i \leq n$ and all $\xi, \zeta \in \mathcal{C}_{\bar{\omega}}$.

Proof. The proof is by induction. For $i = 1$, the Hölder inequality (1.2) and $\xi, \zeta \in \mathcal{C}_{\bar{\omega}}$ imply

$$\|\psi_{\tau^{-1}(\xi)}^{-1}(x) - \psi_{\tau^{-1}(\zeta)}^{-1}(x)\| \leq C_{\Psi} d_{\Sigma_k}(\tau^{-1}(\xi), \tau^{-1}(\zeta))^{\alpha} \leq \nu^{-\alpha} d_{\Sigma_k}(\xi, \zeta)^{\alpha}.$$

We argue inductively. Suppose that the lemma holds for $i - 1$, $i < n$:

$$\|\psi_{\tau^{-1}(\xi)}^{-(i-1)}(x) - \psi_{\tau^{-1}(\zeta)}^{-(i-1)}(x)\| < C_{\Psi} \nu^{-\alpha(i-1)} \sum_{j=0}^{i-2} (\lambda^{-1} \nu^{\alpha})^j d_{\Sigma_k}(\xi, \zeta)^{\alpha}, \quad (4.4)$$

for every $\xi, \zeta \in \mathcal{C}_{\bar{\omega}}$. We will see that the estimate also holds for i . By the triangle inequality,

$$\begin{aligned} \|\psi_{\tau^{-1}(\xi)}^{-i}(x) - \psi_{\tau^{-1}(\zeta)}^{-i}(x)\| &\leq \|\psi_{\tau^{-1}(\xi)}^{-i}(x) - \psi_{\tau^{-1}(\xi)}^{-1} \circ \psi_{\tau^{-1}(\zeta)}^{-(i-1)}(x)\| + \\ &\quad + \|\psi_{\tau^{-1}(\xi)}^{-1} \circ \psi_{\tau^{-1}(\zeta)}^{-(i-1)}(x) - \psi_{\tau^{-1}(\zeta)}^{-i}(x)\|. \end{aligned}$$

Since the inverse of these functions expand at most $1/\lambda$, the above equation is less than or equal to

$$\frac{1}{\lambda} \|\psi_{\tau^{-1}(\xi)}^{-(i-1)}(x) - \psi_{\tau^{-1}(\zeta)}^{-(i-1)}(x)\| + \|\psi_{\tau^{-1}(\xi)}^{-1}(y) - \psi_{\tau^{-1}(\zeta)}^{-1}(y)\|,$$

where $y = \psi_{\tau^{-1}(\zeta)}^{-(i-1)}(x) \in \bar{D}$. By the induction hypothesis (4.4),

$$\frac{1}{\lambda} \|\psi_{\tau^{-1}(\xi)}^{-(i-1)}(x) - \psi_{\tau^{-1}(\zeta)}^{-(i-1)}(x)\| \leq C_{\Psi} \lambda^{-1} \nu^{-\alpha(i-1)} \sum_{j=0}^{i-2} (\lambda^{-1} \nu^{\alpha})^j d_{\Sigma_k}(\xi, \zeta)^{\alpha}.$$

As $y \in \bar{D}$, applying the Hölder inequality (1.2) and since $\xi, \zeta \in \mathcal{C}_{\bar{\omega}}$ we get

$$\|\psi_{\tau^{-1}(\xi)}^{-1}(y) - \psi_{\tau^{-1}(\zeta)}^{-1}(y)\| \leq C_{\Psi} d_{\Sigma_k}(\tau^{-1}(\xi), \tau^{-1}(\zeta))^{\alpha} \leq C_{\Psi} \nu^{-\alpha i} d_{\Sigma_k}(\xi, \zeta)^{\alpha}.$$

Putting together the previous inequalities we obtain

$$\begin{aligned} &C_{\Psi} \lambda^{-1} \nu^{-\alpha(i-1)} \sum_{j=0}^{i-2} (\lambda^{-1} \nu^{\alpha})^j d_{\Sigma_k}(\xi, \zeta)^{\alpha} + C_{\Psi} \nu^{-\alpha i} d_{\Sigma_k}(\xi, \zeta)^{\alpha} \\ &= C_{\Psi} \nu^{-\alpha i} \sum_{j=0}^{i-1} (\lambda^{-1} \nu^{\alpha})^j d_{\Sigma_k}(\xi, \zeta)^{\alpha}, \end{aligned}$$

ending the proof of the lemma. \square

4.2. Proof of Theorem 4.1. Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{S}$ with $\alpha > 0$, $\nu^{\alpha} < \lambda < 1$ and an open set $B \subset D$. Recall that we need to prove the following:

B satisfies the covering property for $\text{IFS}(\Phi) \iff$ there are $\delta > 0$
and a neighborhood \mathcal{V} of Φ in \mathcal{S} such that $\Gamma_{\Psi}^{+}(B) \cap H^s \neq \emptyset$
for every $\Psi \in \mathcal{V}$ and every δ -horizontal disk H^s in $\Sigma_k \times B$.

\Leftarrow We will see that if the covering property is not satisfied then intersection (4.1) is also not satisfied. If B does not satisfy the covering property then there is $x \in \bar{B}$ such that $x \notin \phi_i(B)$ for all $i = 1, \dots, k$. We can assume that $x \in B$, otherwise we can take an arbitrarily small perturbation $\Psi = \tau \ltimes (\psi_1, \dots, \psi_k)$ of Φ such that the covering property in B for $\text{IFS}(\Psi)$

is not satisfied for a point in B . The condition $x \notin \phi_i(B)$ for all $i = 1, \dots, k$ implies that $\Phi^{-1}(\xi, x) \notin \Sigma_k \times \overline{B}$ for all $\xi \in \Sigma_k$ and hence

$$(\xi, x) \notin \bigcap_{n \geq 0} \Phi^n(\Sigma_k \times \overline{B}) = \Gamma_\Phi^+(B) \quad \text{for all } \xi \in \Sigma_k.$$

Therefore $\Gamma_\Phi^+(B)$ does not meet the horizontal disk $H^s = W_{loc}^s(\xi; \tau) \times \{x\}$, and thus the intersection property (4.1) is not verified.

\implies We split the proof of the fact that the covering property implies the intersection condition into two steps.

Choice of the neighborhood \mathcal{V} of Φ . Recall that given an open covering \mathcal{C} of a compact set X of a metric space, there is a constant $L > 0$ and a *Lebesgue number* of \mathcal{C} , such that every subset of X with diameter less than L is contained in some member of \mathcal{C} .

Let $2L > 0$ be a Lebesgue number of the open covering $\{\phi_1(B), \dots, \phi_k(B)\}$ of \overline{B} . Note that there are C^0 -neighborhoods \mathcal{U}_i of ϕ_i such that the family

$$B_i = \text{int} \left(\bigcap_{\psi \in \mathcal{U}_i} \psi(B) \right), \quad i = 1, \dots, k, \quad (4.5)$$

is an open covering of \overline{B} . By shrinking the size of the sets \mathcal{U}_i we can assume that L is a Lebesgue number of this covering. We can also assume that any $\psi \in \mathcal{U}_i$ is a C^0 -(λ, β)-Lipschitz map on \overline{B} for all $i = 1, \dots, k$.

We take a neighborhood \mathcal{V} of Φ in \mathcal{S} such that if $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ then $\psi_\xi \in \mathcal{U}_i$ with $i = \xi_0$. By (4.5),

$$\psi_{\tau^{-1}(\xi)}^{-1}(\overline{B}_i) \subset B \quad \text{for all } \xi \in \Sigma_k \text{ with } \xi_{-1} = i. \quad (4.6)$$

Since Φ is a one-step map then $\phi_\xi = \phi_\zeta$ if $\xi_0 = \zeta_0$, and hence we can take the Hölder constant $C_\Phi = 0$. The definition of the distance in (1.3) implies C_Ψ is close to $C_\Phi = 0$. By hypothesis $\nu^\alpha < \lambda$, and so shrinking the neighborhood \mathcal{V} we can assume that for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$,

$$C_\Psi \sum_{i=0}^{\infty} (\lambda^{-1} \nu^\alpha)^i < L/2. \quad (4.7)$$

This completes the choice of the neighborhood \mathcal{V} of Φ .

Existence of a point in $\Gamma_\Psi^+(B) \cap H^s$. Let us consider \mathcal{V} to be the above neighborhood of Φ . The main step is the following proposition.

Proposition 4.3. *Consider $0 < \delta < \lambda L/2$ and let H^s be a δ -horizontal disk in $\Sigma_k \times B$ associated with some $(\zeta, z) \in \Sigma_k \times B$ and α -Hölder constant $C \geq 0$. Then for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ there are an infinite word $\bar{\omega} = \dots \omega_{-n} \dots \omega_{-1}$ with $\omega_{-n} \in \{1, \dots, k\}$ and a sequence of nested compact subsets $\{V_n\}$ of M such that for every $n \in \mathbb{N}$ it holds that*

$$V_n \subset \mathcal{P}(H^s \cap (\mathcal{C}_{\bar{\omega}_{-n}}(\zeta) \times B)),$$

$$\psi_{\tau^{-1}(\xi)}^{-n}(V_n) \subset B \quad \text{and} \quad \text{diam}(\psi_{\tau^{-1}(\xi)}^{-n}(V_n)) \leq C(\lambda^{-1} \nu^\alpha)^n$$

for all $\xi \in \mathcal{C}_{\bar{\omega}_{-n}}(\zeta)$ where $\bar{\omega}_{-n} = \omega_{-n} \dots \omega_{-1}$.

Let us see how the implication (\implies) follows from this proposition. Let

$$\{x\} = \bigcap_{n \in \mathbb{N}} V_n \subset B \quad \text{and} \quad \{\xi\} = \bigcap_{n \in \mathbb{N}} \mathcal{C}_{\bar{\omega}_{-n}}(\zeta) \subset W_{loc}^s(\zeta; \tau).$$

Observe that $(\xi, x) \in H^s$ and $\psi_{\tau^{-1}(\xi)}^{-n}(x) \in B$ for all $n \in \mathbb{N}$. Thus $\Psi^{-n}(\xi, x) \in \Sigma_k \times B$ for all $n \in \mathbb{N}$ and hence $(\xi, x) \in \Gamma_{\Psi}^+(B)$. Therefore $\Gamma_{\Psi}^+(B) \cap H^s \neq \emptyset$.

To complete the proof of Theorem 4.1 it remains to prove the proposition.

Proof of Proposition 4.3. Fix $\Psi = \tau \ltimes \psi_{\xi} \in \mathcal{V}$ and consider the (α, C) -Hölder map $h : W_{loc}^s(\zeta; \tau) \rightarrow B$ associated with the δ -horizontal disk $H^s \subset \Sigma_k \times B$ (see Definition 1.2). The construction of the nested sequence of sets $\{V_n\}$ and the infinite word $\bar{\omega} = \dots \omega_{-m} \dots \omega_{-1}$ is done inductively. Let

$$V \stackrel{\text{def}}{=} \mathcal{P}(H^s) \subset B.$$

Note that $\text{diam}(V) \leq 2\delta < L$. By the definition of the Lebesgue number, we have that $V \subset B_{i_1}$ for some $i_1 \in \{1, \dots, k\}$. Recall the definition of the relative cylinder in (4.3) associated with $\zeta \in \Sigma_k$ and the word $\bar{\omega}_{-1} = i_1$ and consider the set

$$V_1 \stackrel{\text{def}}{=} \mathcal{P}(H^s \cap (\mathcal{C}_{\bar{\omega}_{-1}}(\zeta) \times V)).$$

By construction $V_1 \subset V \subset B_{i_1}$ and by (4.6),

$$\psi_{\tau^{-1}(\xi)}^{-1}(V_1) \subset B \quad \text{for all } \xi \in \mathcal{C}_{\bar{\omega}_{-1}}(\zeta). \quad (4.8)$$

Claim 4.4. $\text{diam}(V_1) \leq \delta_1 \stackrel{\text{def}}{=} C\nu^\alpha$.

Proof. Given x and y in V_1 there are ξ and η in $\mathcal{C}_{\bar{\omega}_{-1}}(\zeta)$ such that $x = h(\xi)$ and $y = h(\eta)$. Since h is (α, C) -Hölder continuous,

$$\|x - y\| = \|h(\xi) - h(\eta)\| \leq C d_{\Sigma_k}(\xi, \eta)^\alpha \leq C\nu^\alpha = \delta_1,$$

proving the claim. \square

By Claim 4.4 and since the fiber-maps ψ_{ξ} are (λ, β) -Lipschitz on \bar{D} ,

$$\text{diam}(\psi_{\tau^{-1}(\xi)}^{-1}(V_1)) \leq \lambda^{-1}\delta_1 \quad \text{for all } \xi \in \mathcal{C}_{\bar{\omega}_{-1}}(\zeta).$$

Recalling that $C\nu^\alpha < \delta$ (see Definition 1.2) we get that

$$\lambda^{-1}\delta_1 = \lambda^{-1}C\nu^\alpha < \lambda^{-1}\delta \leq L/2.$$

Therefore,

$$\text{diam}(\psi_{\tau^{-1}(\xi)}^{-1}(V_1)) \leq \lambda^{-1}\delta_1 \leq L/2.$$

Arguing inductively, we suppose that we have constructed a finite word $\bar{\omega}_{-n} := \omega_{-n} \dots \omega_{-1}$ (the word $\bar{\omega}_{-i}$ is obtained adding the letter ω_{-i} to the word $\bar{\omega}_{-i+1}$) and closed sets $V_n \subset V_{n-1} \subset \dots \subset V_1$ with

$$\text{diam}(V_n) \leq C\nu^{n\alpha} \stackrel{\text{def}}{=} \delta_n \quad (4.9)$$

and such that for every $\xi \in \mathcal{C}_{\bar{\omega}_{-n}}(\zeta)$,

$$\psi_{\tau^{-1}(\xi)}^{-n}(V_n) \subset B \quad \text{and} \quad \text{diam}(\psi_{\tau^{-1}(\xi)}^{-n}(V_n)) \leq \lambda^{-n}\delta_n. \quad (4.10)$$

We now construct the word $\bar{\omega}_{-(n+1)}$ and the closed set $V_{n+1} \subset V_n$ satisfying analogous inclusions and inequalities. By (4.10) we have that

$$A_n \stackrel{\text{def}}{=} \bigcup_{\xi \in \mathcal{C}_{\bar{\omega}_{-n}}(\zeta)} \psi_{\tau^{-1}(\xi)}^{-n}(V_n) \subset B.$$

Claim 4.5. $\text{diam}(A_n) < L$.

Proof. Given \bar{x} and \bar{y} in A_n there are $x, y \in V_n$ and $\xi, \eta \in \mathcal{C}_{\bar{\omega}_{-n}}(\zeta)$ such that $\bar{x} = \psi_{\tau^{-1}(\xi)}^{-n}(x)$ and $\bar{y} = \psi_{\tau^{-1}(\eta)}^{-n}(y)$. Then

$$\begin{aligned} \|\bar{x} - \bar{y}\| &= \|\psi_{\tau^{-1}(\xi)}^{-n}(x) - \psi_{\tau^{-1}(\eta)}^{-n}(y)\| \\ &\leq \|\psi_{\tau^{-1}(\xi)}^{-n}(x) - \psi_{\tau^{-1}(\eta)}^{-n}(x)\| + \|\psi_{\tau^{-1}(\eta)}^{-n}(x) - \psi_{\tau^{-1}(\eta)}^{-n}(y)\| \\ &\leq C_{\Psi} \nu^{-\alpha n} \sum_{j=0}^{n-1} (\lambda^{-1} \nu^{\alpha})^j d_{\Sigma_k}(\xi, \eta)^{\alpha} + \lambda^{-n} \delta_n \end{aligned}$$

where the last inequality follows from Lemma 4.2 and the induction hypothesis (4.10). Since ξ and η belong to $\mathcal{C}_{\bar{\omega}_{-n}}(\zeta)$ then $d_{\Sigma_k}(\xi, \eta)^{\alpha} \leq \nu^{\alpha n}$. Hence

$$\|\bar{x} - \bar{y}\| \leq C_{\Psi} \sum_{j=0}^{n-1} (\lambda^{-1} \nu^{\alpha})^j + \lambda^{-n} \delta_n \leq L/2 + \lambda^{-n} \delta_n$$

where the last inequality follows from (4.7). Note that

$$\lambda^{-n} \delta_n = C (\lambda^{-1} \nu^{\alpha})^n \leq C \lambda^{-1} \nu^{\alpha} < \lambda^{-1} \delta < L/2.$$

Therefore for every pair of points $\bar{x}, \bar{y} \in A_n$ we have $\|\bar{x} - \bar{y}\| < L$ and thus $\text{diam}(A_n) < L$, proving the claim. \square

Since L is a Lebesgue number of the covering $\{B_i\}_{i=1}^k$, the claim implies there is $i_{n+1} \in \{1, \dots, k\}$ such that $A_n \subset B_{i_{n+1}}$. We let

$$\bar{\omega}_{-(n+1)} = i_{n+1} \omega_{-n} \dots \omega_{-1} \quad \text{and} \quad V_{n+1} = \mathcal{P}(H^s \cap (\mathcal{C}_{\bar{\omega}_{-(n+1)}}(\zeta) \times V_n)).$$

Note that by construction $V_{n+1} \subset V_n$.

Claim 4.6. $\text{diam}(V_{n+1}) \leq C \nu^{(n+1)\alpha} \stackrel{\text{def}}{=} \delta_{n+1}$.

Proof. Given $x, y \in V_{n+1}$ there are $\xi, \eta \in \mathcal{C}_{\bar{\omega}_{-(n+1)}}(\zeta)$ such that $x = h(\xi)$ and $y = h(\eta)$. From the (α, C) -Hölder continuity of h and since $\xi, \eta \in \mathcal{C}_{\bar{\omega}_{-(n+1)}}(\zeta)$ we get $\|x - y\| \leq C d_{\Sigma_k}(\xi, \eta)^{\alpha} \leq C \nu^{(n+1)\alpha}$, concluding the proof. \square

Using that $V_{n+1} \subset V_n$, $\text{diam}(V_n) \leq \delta_{n+1}$, and equations (4.6) and (4.10) we get

$$\psi_{\tau^{-1}(\xi)}^{-(n+1)}(V_{n+1}) \subset B \quad \text{and} \quad \text{diam}(\psi_{\tau^{-1}(\xi)}^{-(n+1)}(V_{n+1})) \leq \lambda^{-(n+1)} \delta_{n+1},$$

for every $\xi \in \mathcal{C}_{\bar{\omega}_{-(n+1)}}(\zeta)$. Therefore (4.10) holds for $(n+1)$ -step and we can continue arguing inductively. This completes the construction of the infinite word $\bar{\omega}$ and the sequence of nested sets $\{V_n\}$ in the proposition, ending the proof. \square

The proof of Theorem 4.1 is now complete.

4.3. Partially hyperbolic symbolic skew-products. For applications of blenders we would need to have additional fiber maps, that are not necessarily contracting or forward invariant in the region D . Thus, we will embed a symbolic blender for a one-step map Φ defined on $\Sigma_k \times M$ into another one-step map $\hat{\Phi}$ on $\Sigma_d \times M$ with $d \geq k$. Firstly, we will impose some partially hyperbolic conditions on $\hat{\Phi}$, which are motivated by the reduction of partial hyperbolic diffeomorphisms to the setting of symbolic skew-products (see next section, Proposition 5.2).

Definition 4.7 (Partial hyperbolic symbolic skew-products). *A symbolic skew-product $\hat{\Phi} \in \mathcal{S}_{d,\gamma,\hat{\gamma}^{-1}}^{r,\alpha}(M)$ is called partial hyperbolic if*

$$\nu^\alpha < \gamma < 1 < \hat{\gamma}^{-1} < \nu^{-\alpha}.$$

We will denote by $\mathcal{PHS}_d^{r,\alpha}(M)$ the set of partial hyperbolic symbolic α -Hölder skew-products on $\Sigma_d \times M$ with C^r -fiber maps.

Under the assumption of these partially hyperbolic conditions, the usual graph transform argument yields a local strong stable and unstable laminations for a symbolic skew-product $\hat{\Phi}$.

Proposition 4.8 ([AV10]). *Let $\hat{\Phi} \in \mathcal{S}_{d,\gamma,\hat{\gamma}^{-1}}^{0,\alpha}(M)$ be a symbolic skew-product satisfying the s -domination condition $\nu^\alpha < \gamma$. Then, there exists a partition*

$$\mathcal{W}_\Phi^s = \{W_{loc}^{ss}((\xi, x); \hat{\Phi}) : (\xi, x) \in \Sigma_d \times M\}$$

of $\Sigma_d \times M$ such that denoting $C = C_{\hat{\Phi}}(1 - \gamma^{-1}\nu^\alpha)^{-1} \geq 0$, it holds

i) every leaf $W_{loc}^{ss}((\xi, x); \hat{\Phi})$ is the graph of an α -Hölder function

$$\gamma_{\xi,x,\hat{\Phi}}^s : W_{loc}^s(\xi; \tau) \rightarrow M$$

with α -Hölder constant less or equal than C (uniform on ξ and x), and it depends continuously on $\hat{\Phi}$,

ii) $W_{loc}^{ss}((\xi, x); \hat{\Phi})$ varies continuously with respect to (ξ, x) , i.e., the map

$$(\xi, \xi', x) \mapsto \gamma_{\xi,x,\hat{\Phi}}^s(\xi') \quad \text{is continuous}$$

where (ξ, ξ') varies in the space of pairs of points in the same local stable set for τ ,

iii) $\hat{\Phi}(W_{loc}^{ss}((\xi, x); \hat{\Phi})) \subset W_{loc}^{ss}(\hat{\Phi}(\xi, x); \hat{\Phi})$ for all $(\xi, x) \in \Sigma_d \times M$,

iv) $W_{loc}^{ss}((\xi, x); \hat{\Phi}) \subset W^s((\xi, x); \hat{\Phi})$.

Here $W^s((\xi, x); \hat{\Phi}) \stackrel{\text{def}}{=} \{(\zeta, y) \in \Sigma_d \times M : \lim_{n \rightarrow \infty} d(\hat{\Phi}^n(\zeta, y), \hat{\Phi}^n(\xi, x)) = 0\}$.

There is a dual statement of Proposition 4.8 for symbolic skew-product $\hat{\Phi}$ satisfying the u -dominating condition $\hat{\gamma}^{-1} < \nu^{-\alpha}$. Namely, one has a partition

$$\mathcal{W}_\Phi^u = \{W_{loc}^{uu}((\xi, x); \hat{\Phi}) : (\xi, x) \in \Sigma_d \times M\}$$

of $\Sigma_d \times M$ verifying analogous properties. Each leaf $W_{loc}^{uu}((\xi, x); \hat{\Phi})$ of this partition \mathcal{W}_Φ^u is said to be *local strong unstable set* throughout the point (ξ, x) . We define the *global strong unstable set* of (ξ, x) as

$$W^{uu}((\xi, x); \hat{\Phi}) \stackrel{\text{def}}{=} \bigcup_{n \geq 0} \Phi^n(W_{loc}^{uu}(\hat{\Phi}^{-n}(\xi, x); \hat{\Phi}))$$

For a $\hat{\Phi}$ -invariant set Γ we define *local strong unstable set* of Γ for $\hat{\Phi}$ as

$$W_{loc}^{uu}(\Gamma; \hat{\Phi}) = \bigcup_{(\xi, x) \in \Gamma} W_{loc}^{uu}((\xi, x); \hat{\Phi}).$$

In the same manner, the *global strong unstable set* of Γ , denoted by $W^{uu}(\Gamma; \hat{\Phi})$, is defined as the union of the global strong unstable sets of the points in Γ and it holds

$$W_{loc}^{uu}(\Gamma; \hat{\Phi}) \subset W^{uu}(\Gamma; \hat{\Phi}) \subset W^u(\Gamma; \hat{\Phi}).$$

The *local/global strong stable sets* are defined similarly and satisfy analogous inclusions.

Proposition 4.9 (Embedded blender). *Let $\hat{\Phi} \in \mathcal{PHS}_d^{0,\alpha}(M)$ be a one-step skew-product map with $d \geq k$. Assume that the restriction*

$$\Phi = \hat{\Phi}|_{\Sigma_k \times M} = \tau \ltimes (\phi_1, \dots, \phi_k)$$

has a symbolic cs-blender $\Gamma_{\Phi}^{cs} \subset \Sigma_k \times M$ whose superposition domain contains an open set $B \subset M$ satisfying that $\bar{B} \subset \phi_1(B) \cup \dots \cup \phi_k(B)$. Then there exists $\delta > 0$ such that for every small enough $\mathcal{S}^{0,\alpha}$ -perturbation $\hat{\Psi}$ of $\hat{\Phi}$ it holds

$$W_{loc}^{uu}(\Gamma_{\Psi}^{cs}; \hat{\Psi}) \cap H^s \neq \emptyset \quad \text{for every } \delta\text{-horizontal disk } H^s \text{ in } \Sigma_d \times B$$

where Γ_{Ψ}^{cs} is the continuation of Γ_{Φ}^{cs} for $\Psi = \hat{\Psi}|_{\Sigma_k \times M}$.

Proof. Since Γ_{Φ}^{cs} is a symbolic blender for Φ whose superposition domain contains the open set B then $\Gamma_{\Phi}^+(B) \subset \Gamma_{\Phi}^{cs}$. Recall that $\Gamma_{\Phi}^+(B)$ is the forward maximal invariant set in $\Sigma_k \times B$ for Φ . We can enlarge the size of B a little, call this new set \tilde{B} , assuming that for every small enough $\mathcal{S}^{0,\alpha}$ -perturbation $\hat{\Psi}$ of $\hat{\Phi}$, it holds

$$\Gamma_{\Psi}^+(\tilde{B}) \subset \Gamma_{\Psi}^{cs}, \quad \text{where } \Psi = \hat{\Psi}|_{\Sigma_k \times M}. \quad (4.11)$$

Observe that for each $(\xi, x) \in \Sigma_d \times M$, since $\hat{\Phi}$ is a one-step map then

$$W_{loc}^{uu}((\xi, x); \hat{\Phi}) = W_{loc}^u(\xi; \tau) \times \{x\}.$$

Since the strong unstable set depends continuously on $\hat{\Phi}$, we can assume that for every small enough $\mathcal{S}^{0,\alpha}$ -perturbation $\hat{\Psi}$ of $\hat{\Phi}$,

$$W_{loc}^{uu}((\xi, x); \hat{\Psi}) \subset \Sigma_d \times \tilde{B} \quad \text{for all } (\xi, x) \in \Sigma_d \times B. \quad (4.12)$$

A slight modification of the proof of Theorem 4.1 shows that there is $\delta > 0$ such that for every small enough $\mathcal{S}^{0,\alpha}$ -perturbation $\hat{\Psi}$ of $\hat{\Phi}$ we have

$$\Gamma_{\hat{\Psi}}^+(\Sigma_{k,d}^- \times B) \cap H^s \neq \emptyset \quad (4.13)$$

for every δ -horizontal disk H^s in $\Sigma_d \times B$. Here

$$\Gamma_{\hat{\Psi}}^+(\Sigma_{k,d}^- \times B) = \bigcap_{n \geq 0} \hat{\Psi}^n(\Sigma_{k,d}^- \times B)$$

and $\Sigma_{k,d}^- = \{\xi = (\xi_i)_{i \in \mathbb{Z}} \in \Sigma_d : \xi_i \in \{1, \dots, k\} \text{ for } i < 0\}$. This assertion is showed by repeating the same argument as in the proof of Theorem 4.1, using the s -domination condition $\nu^\alpha < \gamma$ (instead of $\nu^\alpha < \lambda$) and the global Hölder continuity of the fiber maps with respect to the base point.

Consider a $\mathcal{S}^{0,\alpha}$ -perturbation $\hat{\Psi} = \tau \ltimes \psi_\xi$ of $\hat{\Phi}$ satisfying (4.11), (4.12) and (4.13) and let $\Gamma_{\Psi}^{cs} \subset \Sigma_k \times M$ be the continuation of Γ_{Φ}^{cs} for $\Psi = \hat{\Psi}|_{\Sigma_k \times M}$.

Claim 4.10. $\Gamma_{\hat{\Psi}}^+(\Sigma_{k,d}^- \times B) \subset W_{loc}^{uu}(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi})$.

Proof. Fix a point $(\xi, x) \in \Gamma_{\hat{\Psi}}^+(\Sigma_{k,d}^- \times \tilde{B})$ we show that

$$W_{loc}^{uu}((\xi, x); \hat{\Psi}) \cap (\Sigma_k \times \tilde{B}) \subset \Gamma_{\hat{\Psi}}^{cs}. \quad (4.14)$$

This assertion proves the claim since if $(\zeta, z) \in W_{loc}^{uu}((\xi, x); \hat{\Psi}) \cap (\Sigma_k \times \tilde{B})$ then $(\xi, x) \in W_{loc}^{uu}((\zeta, z); \hat{\Psi}) \subset W_{loc}^{uu}(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi})$.

Let $(\zeta, z) \in W_{loc}^{uu}((\xi, x); \hat{\Psi}) \cap (\Sigma_k \times \tilde{B})$. In order to prove (4.14), observe that from (4.11) it is enough to show that $\hat{\Psi}^{-n}(\zeta, z) = \Psi^{-n}(\zeta, z) \in \Sigma_k \times \tilde{B}$ for all $n \geq 0$. From the invariance of the local strong unstable lamination (dual statement of Item (iii) in Proposition 4.8) it follows that

$$\hat{\Psi}^{-n}(\zeta, z) \in \hat{\Psi}^{-n}(W_{loc}^{uu}((\xi, x); \hat{\Psi})) \subset W_{loc}^{uu}(\hat{\Psi}^{-n}(\xi, x); \hat{\Psi}) \quad (4.15)$$

for all $n \geq 0$. Since $(\xi, x) \in \Gamma_{\hat{\Psi}}^+(\Sigma_{k,d}^- \times B)$ then $\hat{\Psi}^{-n}(\xi, x) \in \Sigma_d \times B$ for all $n \geq 0$ and according to (4.12) it follows that

$$W_{loc}^{uu}(\hat{\Psi}^{-n}(\xi, x); \hat{\Psi}) \subset \Sigma_d \times \tilde{B} \quad \text{for all } n \geq 0. \quad (4.16)$$

Hence, since $\zeta \in \Sigma_k$, (4.15) and (4.16) imply that $\hat{\Psi}^{-n}(\zeta, z) \in \Sigma_k \times \tilde{B}$ for all $n \geq 0$. Therefore $(\zeta, z) \in \Gamma_{\hat{\Psi}}^+(\tilde{B}) \subset \Gamma_{\hat{\Psi}}^{cs}$ concluding (4.14) and the proof of the claim. \square

To conclude the proposition it suffices to observe that the above claim and (4.13) imply that $W_{loc}^{uu}(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi}) \cap H^s \neq \emptyset$ for every δ -horizontal disk H^s in $\Sigma_d \times B$. The proof is now complete. \square

Observe that the set $W_{loc}^s(\zeta; \tau) \times \{z\}$ is an horizontal disk and in the one-step case this set coincides to the local strong stable set of (ζ, z) . For Hölder perturbations of one-step maps, Proposition 4.8 implies that the local strong stable sets are almost horizontal disks. Then, we obtain the following remark:

Remark 4.11. Let $\hat{\Phi} \in \mathcal{PHS}_d^{0,\alpha}(M)$ be an one-step skew-product map as in the hypothesis of Proposition 4.9. For every small enough $\mathcal{S}^{0,\alpha}$ -perturbation $\hat{\Psi}$ of $\hat{\Phi}$ and every $(\xi, x) \in \Sigma_d \times B$ there exists $(\zeta, z) \in \Gamma_{\hat{\Psi}}^{cs}$ such that

$$W_{loc}^{uu}((\zeta, z); \hat{\Psi}) \cap W_{loc}^{ss}((\xi, x); \hat{\Psi}) \neq \emptyset.$$

5. APPLICATIONS: ROBUST HETERODIMENSIONAL CYCLES AND MIXING SETS

Our goal here is to generate robust heterodimensional cycles (Section 5.2) and robust non-hyperbolic mixing sets (Section 5.3) in the presence of symbolic blenders. We will start by explaining how to go from partially hyperbolic diffeomorphisms to symbolic skew-products. In this section, N and M denote compact Riemannian manifolds and $\text{id}: M \rightarrow M$ the identity map.

5.1. Reduction to perturbations of symbolic skew-products. Let F be a C^2 -diffeomorphism of N with a horseshoe $\Lambda \subset N$ of d -legs which has a DF -invariant splitting of the tangent bundle

$$T_\Lambda N = E_\Lambda^s \oplus E_\Lambda^u$$

such that in a Riemannian metric on N there is $0 < \mu < \nu < 1$ satisfying

$$\mu \leq \|D_z F(v)\| \leq \nu \quad \text{and} \quad \mu \leq \|D_z F^{-1}(w)\| \leq \nu$$

for all unit vectors $v \in E_z^s$, $w \in E_z^u$ and $z \in \Lambda$. We assume that E_Λ^s , E_Λ^u are trivial product bundles. Both, the above assumption and the extra regularity of F , are technical conditions necessary for the main result in [IN12], but it is conjectured that it holds without them (see also [Gor06, PSW12]). This result is our main tool to prove the next proposition (see Proposition 5.2).

Let f be a C^1 skew-product diffeomorphism over F defined as

$$f : N \times M \rightarrow N \times M, \quad f(z, x) = (F(z), \phi(z, x)) \quad (5.1)$$

where $\phi(z, \cdot) : M \rightarrow M$ is a family of C^1 -diffeomorphisms such that there are positive numbers γ and $\hat{\gamma}$ satisfying

$$\gamma < \|D_z \phi(z, \cdot)v\| < \hat{\gamma}^{-1} \quad \text{for all unit vector } v \in T_z M \text{ and } z \in \Lambda. \quad (5.2)$$

We will assume that the skew-product diffeomorphism (5.1) satisfies that

$$\nu + L_f < \nu^{-1} \quad \text{and} \quad \nu < \gamma < \hat{\gamma}^{-1} < \nu^{-1} \quad (5.3)$$

where for a fixed $\delta > 0$ small enough

$$L_f = \sup_{x \in M} \left\{ \frac{\|\phi^{\pm 1}(z, x) - \phi^{\pm 1}(z', x)\|}{d_N(z, z')} : z, z' \in \Lambda, 0 < d_N(z, z') < \delta \right\} \geq 0.$$

We say that L_f is the *local Lipschitz constant* of f (or f^{-1}). The inequalities (5.3) are called *modified dominated splitting condition* in [IN12]. The first inequality is clearly verified if $L_f = 0$. The other inequalities are the dominated conditions in the definition of partial hyperbolicity.

Definition 5.1. A skew-product diffeomorphism f as (5.1) is said to be locally constant if $L_f = 0$.

Partial hyperbolic locally constant skew-product diffeomorphisms over a horseshoe satisfy the modified dominated splitting condition (5.3). Modified dominated splitting condition is a C^1 -open condition since the same property is satisfied for any diffeomorphism C^1 -close. However, a C^1 -close diffeomorphism g of f is a priori not a skew-product. Under the assumptions (5.3), from Hirsch-Pugh-Shub theory [HPS77] or from the recent works of [Gor06, IN12, PSW12] it follows that g is topologically conjugated to a skew-product. We will explain more about this.

Proposition 5.2. Let f be a C^1 -skew-product diffeomorphism as (5.1) satisfying (5.2) and (5.3). Then given $\varepsilon > 0$ small enough, any ε -perturbation g of f in the C^1 -topology has a locally maximal invariant set $\Delta \subset N \times M$ homeomorphic to $\Lambda \times M$ such that $g|_\Delta$ is conjugated to a symbolic skew-product

$$\Psi_g \in \mathcal{PHS}_d^{1,\alpha}(M)$$

with $\alpha = \log \nu / \log \mu > 0$ and local Hölder constant $C_{\Psi_g} = L_f + O(\varepsilon)$.

Remark 5.3. In the special case of $f = F \times \text{id}$, from [Gor06, Theorem B], the above Proposition 5.2 holds without both technical assumptions, the extra regularity of the base map F and the trivialization of the vector bundles.

Proof of Proposition 5.2. For each $z \in \Lambda$ we consider the fiber $\mathcal{L}_z = \{z\} \times M$. The collection \mathcal{L} of these fibers is an invariant lamination of f . In [HPS77, Theorems 7.1] and also in [IN12, Theorem A] and [PSW12, Theorem C] it is showed that this lamination is C^1 -persistent. The C^1 -persistence of such lamination means that for any C^1 -perturbation of f , there exists a lamination, C^1 -close to \mathcal{L} , which is preserved by the new dynamics, and such that the dynamics induced on the space of the leaves remains the same.

Given $\varepsilon > 0$ small enough we take g a C^1 -diffeomorphism ε -close to f in the C^1 -topology. Note that,

$$g(z, x) = (\tilde{F}(z, x), \tilde{\phi}(z, x)), \quad (z, x) \in N \times M$$

where $\tilde{\phi}(z, \cdot) : M \rightarrow M$ is also a C^1 -diffeomorphism such that

$$\gamma \|x - x'\| < \|\tilde{\phi}(z, x) - \tilde{\phi}(z, x')\| < \hat{\gamma}^{-1} \|z - x'\| \quad (5.4)$$

for all $x, x' \in M$ and z in a neighborhood of Λ .

For each $z \in \Lambda$, the fiber $\mathcal{W}_{\sigma(z)}$, continuation of \mathcal{L}_z for g is parametrized by the graph of a C^1 -map $Q(z, \cdot) : M \rightarrow M \times N$. According to [IN12, Theorem A] and since g is ε -close to f , we get

$$d_N(Q(z, x), Q(z, x')) \leq O(\varepsilon) \|x - x'\|, \quad (5.5)$$

$$d_{C^0}(Q(z, \cdot), z) \leq O(\varepsilon). \quad (5.6)$$

For C^1 maps, the C^0 norm of the first derivative is equal the best Lipschitz constant. Hence, the above inequalities show that $d_{C^1}(Q(z, \cdot), z) \leq O(\varepsilon)$.

Let

$$\Delta = \bigcup_{z \in \Lambda} \mathcal{W}_{\sigma(z)} \subset M \times N.$$

By sending $\mathcal{W}_{\sigma(z)}$ to z we may define a continuous projection $P : \Delta \rightarrow N$ such that $P(\Delta) = \Lambda$ and $F|_{\Lambda} \circ P = P \circ g$. Moreover,

$$h : \Delta \rightarrow \Lambda \times M, \quad h(z, x) = (P(z, x), x)$$

is an homeomorphism whose inverse is $h^{-1}(z, x) = (Q(z, x), x)$. Let

$$\tilde{g} = h \circ g|_{\Delta} \circ h^{-1} : \Lambda \times M \rightarrow \Lambda \times M.$$

Observe that

$$\begin{aligned} \tilde{g}(z, x) &= (P \circ g \circ h^{-1}(z, x), \tilde{\phi} \circ h^{-1}(z, x)) \\ &= (F|_{\Lambda} \circ P \circ h^{-1}(z, x), \tilde{\phi}(Q(z, x), x)) = (F(z), \psi(z, x)). \end{aligned}$$

Thus, \tilde{g} is a skew-product diffeomorphism defined on $\Lambda \times M$ which is conjugated to $g|_{\Delta}$ by means of the conjugation h . Since for each $z \in \Lambda$ the map $\psi(z, \cdot)$ is a composition of C^1 -maps, then it is a C^1 -map. In addition, we can easily check that the rate of contraction and expansion of these maps are uniformly close to $\hat{\gamma}^{-1}$ and γ respectively. Indeed,

$$\begin{aligned} \|\psi(z, x) - \psi(z, x')\| &\leq \\ &\leq \|\tilde{\phi}(Q(z, x), x) - \tilde{\phi}(Q(z, x), x')\| + \|\tilde{\phi}(Q(z, x), x') - \tilde{\phi}(Q(z, x'), x')\|. \end{aligned}$$

By means of (5.4) and (5.5) it follows that

$$\|\psi(z, x) - \psi(z, x')\| < \hat{\gamma}^{-1}\|x - x'\| + O(\varepsilon)\|x - x'\| \leq (\hat{\gamma}^{-1} + O(\varepsilon))\|x - x'\|.$$

In the same way one obtains that

$$\|\psi(z, x) - \psi(z, x')\| \geq |\gamma - O(\varepsilon)|\|x - x'\|.$$

Taking $\varepsilon > 0$ small enough we can assume $\hat{\gamma}^{-1}$ and γ remain, respectively, the rates of contraction and expansion for $\psi(z, \cdot)$. This calculation shows that the derivatives of $\psi(z, \cdot)$ and $\phi(z, \cdot)$ are $O(\varepsilon)$ -close. Moreover, since $Q(z, \cdot)$ is C^1 -close to the constant function $x \mapsto z$, it follows that $\psi(z, \cdot)$ and $\phi(z, \cdot)$ are $O(\varepsilon)$ -close. Consequently $d_{C^1}(\psi(z, \cdot), \phi(z, \cdot)) \leq O(\varepsilon)$.

Although for each $z \in \Lambda$ the maps $\psi(z, \cdot)$ are C^1 -diffeomorphisms, the map \tilde{g} is not necessarily a C^1 -diffeomorphism since h is not a C^1 -conjugation. However, according to [IN12, Theorem A] it follows that both h and h^{-1} are locally α -Hölder continuous maps with $\alpha = \log \nu / \log \mu > 0$. Thus, the function $\psi = \pi_M \circ g \circ h^{-1}$ is locally α -Hölder continuous with respect to the base points: if $d_N(z, z') < \delta$ it holds that

$$d_{C^0}(\psi(z, \cdot), \psi(z', \cdot)) \leq L_g d_{C^0}(h^{-1}(z, \cdot), h^{-1}(z', \cdot)) \leq L_g C_h d_N(z, z')^\alpha$$

where L_g and C_h are the local Lipschitz and α -Hölder constants of g and h respectively.

Recall that $F|_\Lambda$ is conjugated to the Bernoulli shift $\tau : \Sigma_d \rightarrow \Sigma_d$. Let $\ell : \Sigma_d \rightarrow \Lambda$ be the topological conjugation: $\tau = \ell^{-1} \circ F|_\Lambda \circ \ell$. Consider Σ_d with the metric given in (1.5), then from [Gor06, Theorem 2.3] and [BGMV03] it follows that ℓ is a Lipschitz map. That is $d_M(\ell(\xi), \ell(\xi')) \leq L_\ell d_{\Sigma_d}(\xi, \xi')$.

Therefore, we obtain that \tilde{g} is conjugated to a locally α -Hölder symbolic skew-product map

$$\Psi_g : \Sigma_d \times M \rightarrow \Sigma_d \times M, \quad \Psi_g(\xi, x) = (\tau(\xi), \psi_\xi(x)),$$

where $\psi_\xi = \psi(\ell(\xi), \cdot) : M \rightarrow M$ is a $(\gamma, \hat{\gamma}^{-1})$ -Lipschitz C^1 -diffeomorphism with the local Hölder constant $C_{\Psi_g} = L_g C_h L_\ell^\alpha \geq 0$ and $\alpha = \log \nu / \log \mu > 0$.

The same arguments work for a small perturbation g^{-1} of f^{-1} and therefore the inverse map Ψ_g^{-1} is also a skew-product with the same Hölder exponent $0 < \alpha \leq 1$ and local Hölder constant $C_\Psi \geq 0$.

To end the proof we need to show that $C_{\Psi_g} = L_f + O(\varepsilon)$. To do this note that L_g is the local Lipschitz constant of g (or g^{-1}) i.e.,

$$L_g = \sup \left\{ \frac{d(g(z, x), g(z', x'))}{d((z, x), (z', x'))} : (z, x), (z', x') \in \Delta, d_N(z, z') < \delta \right\}$$

and then $L_g = L_f + O(\varepsilon)$. Also, the local Hölder constant C_h of h (or h^{-1}) varies continuously with h that in turn depends continuously on g . In fact, in view of (5.6) we get that h and h^{-1} are close to the identity and so $C_h L_\ell^\alpha = 1 + O(\varepsilon)$. Thus, $C_{\Psi_g} = L_g C_h L_\ell^\alpha = L_f + O(\varepsilon)$, which concludes the proof of the proposition. \square

Note that the above result also holds if f is only defined locally as a skew-product diffeomorphism over the rectangles coming from a Markov partition of Λ . This fact will be useful to define locally constant skew-product diffeomorphisms in the proof of Theorem B and Theorem C.

5.2. Robust heterodimensional cycles: Proof of Theorem B. In this section we will prove theorem B. We will first describe how to build robust cycles in the symbolic setting and then transfer this to dynamics of diffeomorphisms on manifolds, thus proving Theorem B.

Definition 5.4 (Symbolic cycles). *Let $\hat{\Phi} \in \mathcal{PHS}_d^{r,\alpha}(M)$ be a symbolic skew-product with a pair of $\hat{\Phi}$ -invariant sets Γ_1 and Γ_2 . We say that $\hat{\Phi}$ has a symbolic cycle associated with Γ_1 and Γ_2 if their stable/unstable sets meet cyclically, that is, if*

$$W^s(\Gamma_1; \hat{\Phi}) \cap W^u(\Gamma_2; \hat{\Phi}) \neq \emptyset \quad \text{and} \quad W^u(\Gamma_1; \hat{\Phi}) \cap W^s(\Gamma_2; \hat{\Phi}) \neq \emptyset.$$

Assuming that Γ_1 and Γ_2 have continuations, we say that the symbolic cycle is $\mathcal{S}^{r,\alpha}$ -robust if the cyclic intersection of stable/unstable sets holds for every small enough $\mathcal{S}^{r,\alpha}$ -perturbation of $\hat{\Phi}$.

The next proposition shows how to construct robust symbolic cycles using blenders that come from the covering property.

Theorem 5.5 (Symbolic cycles from blenders). *Let $\phi_1, \dots, \phi_k, \phi_{k+1}, \phi_{k+2}$ be $(\gamma, \hat{\gamma}^{-1})$ -Lipschitz C^1 -diffeomorphisms on M and consider disjoint open sets $D_{cs}, D_{cu} \subset M$ such that*

$$\Phi = \tau \times (\phi_1, \dots, \phi_k) \in \mathcal{S}_{k, \lambda_{cs}, \beta_{cs}}^{1,\alpha}(D_{cs}) \cap \mathcal{S}_{k, \lambda_{cu}, \beta_{cu}}^{1,\alpha}(D_{cu})$$

where

$$\nu^\alpha < \gamma \leq \lambda_{cs} < \beta_{cs} < 1 < \lambda_{cu} < \beta_{cu} \leq \hat{\gamma}^{-1} < \nu^{-\alpha}.$$

Assume that there are open subsets $B_{cs} \subset D_{cs}$ and $B_{cu} \subset D_{cu}$ such that the following properties hold:

- *Covering property:*

$$\overline{B_{cs}} \subset \bigcup_{i=1}^k \phi_i(B_{cs}) \quad \text{and} \quad \overline{B_{cu}} \subset \bigcup_{i=1}^k \phi_i^{-1}(B_{cu})$$

- *Cyclic intersections: there exist $x \in B_{cs}$, $y \in D_{cu}$ and $m, n > 0$ such that*

$$\phi_{k+1}^n(x) \in B_{cu} \quad \text{and} \quad \phi_{k+2}^m(y) \in D_{cs}.$$

Then the one-step $\hat{\Phi} = \tau \times (\phi_1, \dots, \phi_k, \phi_{k+1}, \phi_{k+2}) \in \mathcal{PHS}_{k+2}^{1,\alpha}(M)$ has a $\mathcal{S}^{1,\alpha}$ -robust symbolic cycle associated with symbolic blenders whose superposition domains contain B_{cs} and B_{cu} .

Proof. We split the proof of the proposition into two lemmas. Firstly, we need to choose the $\mathcal{S}^{1,\alpha}$ -neighborhood \mathcal{V} of $\hat{\Phi} = \tau \times (\phi_1, \dots, \phi_k, \phi_{k+1}, \phi_{k+2})$ in which we will work. This neighborhood is taken as the intersection of the following three sets:

- \mathcal{V}_1 is the neighborhood of $\hat{\Phi}$ given by Proposition 4.9.
- \mathcal{V}_2 is the neighborhood of $\hat{\Phi}$ such that for every $\hat{\Psi} \in \mathcal{V}_2$ and $y \in D_{cu}$ (from the cyclic intersection hypothesis) it holds

$$\hat{\Psi}^m(\mathcal{C}^+ \times \{y\}) \subset \Sigma_{k+2} \times D_{cs}$$

where \mathcal{C}^+ is the set of the bi-sequence $\xi = (\xi_i)_{i \in \mathbb{Z}} \in \Sigma_{k+2}$ such that $\xi_i = k+2$ for $0 \leq i \leq m$ and $\xi_i \in \{1, \dots, k\}$ otherwise.

- \mathcal{V}_3 is the neighborhood of $\hat{\Phi}$ taken so that the strong stable sets are δ -almost horizontal disks, where $\delta > 0$ is defined in the following manner. Suppose $z \in M$ with $\|z - \phi_{k+1}^n(x)\| < \delta$, then for every $\hat{\Psi} \in \mathcal{V}_3$ it holds

$$\hat{\Psi}^{-n}(\mathcal{C}^- \times \{z\}) \subset \Sigma_{k+2} \times B_{cs}$$

where \mathcal{C}^- is the set of the bi-sequences $\xi = (\xi_i)_{i \in \mathbb{Z}} \in \Sigma_{k+2}$ such that $\xi_i = k+2$ for $-n \leq i < 0$ and $\xi_i \in \{1, \dots, k\}$ otherwise.

Fix $\hat{\Psi} \in \mathcal{V}$, let us to prove that

$$W^s(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi}) \cap W^u(\Gamma_{\hat{\Psi}}^{cu}; \hat{\Psi}) \neq \emptyset.$$

where $\Gamma_{\hat{\Psi}}^{cs}, \Gamma_{\hat{\Psi}}^{cu}$ are the maximal invariant sets in $\Sigma_d \times \overline{D_{cs}}$ and $\Sigma_d \times \overline{D_{cu}}$ of $\hat{\Psi}$ respectively. Note that $\Gamma_{\hat{\Psi}}^{cs}$ and $\Gamma_{\hat{\Psi}}^{cu}$ coincide, respectively, with the maximal invariant set in $\Sigma_k \times \overline{D_{cs}}$ and $\Sigma_k \times \overline{D_{cu}}$ for $\Psi = \hat{\Psi}|_{\Sigma_k \times M}$. Hence, from the covering property assumption and Theorem A, these two $\hat{\Psi}$ -invariant sets are symbolic blenders for Ψ whose superposition domains contain B_{cs} and B_{cu} respectively.

We observe that in the next lemma we do not use the blender property but rather that D_{cs} (D_{cu}) are forward (backward) invariant open sets for contractive (expanding) fiber maps of Φ and its perturbations.

Lemma 5.6. *It holds that $\mathcal{C}^+ \times \{y\} \subset W^s(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi}) \cap W^u(\Gamma_{\hat{\Psi}}^{cu}; \hat{\Psi})$.*

Proof. To prove this lemma we need the following claim. Here $\Sigma_{k,k+2}^+$ represents the bi-sequences $\xi = (\xi_i)_{i \in \mathbb{Z}} \in \Sigma_{k+2}$ such that $\xi_i \in \{1, \dots, k\}$ for $i \geq 0$, and $\Sigma_{k,k+2}^-$ the bi-sequences such that $\xi_i \in \{1, \dots, k\}$ for $i < 0$.

Claim 5.7. $\Sigma_{k,k+2}^+ \times D_{cs} \subset W^s(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi})$ and $\Sigma_{k,k+2}^- \times D_{cu} \subset W^u(\Gamma_{\hat{\Psi}}^{cu}; \hat{\Psi})$.

Proof of the claim. We will prove the first inclusion and the second inclusion is analogous. From Theorem 2.9, the set $\Gamma_{\hat{\Psi}}^{cs}$ is an attractive graph and so

$$\lim_{n \rightarrow \infty} d(\hat{\Psi}^n(\zeta, z), \Gamma_{\hat{\Psi}}^{cs}) = 0, \quad \text{for all } (\zeta, z) \in \Sigma_k \times D_{cs}. \quad (5.7)$$

Thus if $(\xi, p) \in \Sigma_{k,k+2}^+ \times D_{cs}$ then

$$\emptyset \neq W_{loc}^{ss}((\xi, p); \hat{\Psi}) \cap (\Sigma_k \times D_{cs}) \subset W^s(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi}).$$

Now, using the triangular inequality it holds

$$d(\hat{\Psi}^n(\xi, p), \Gamma_{\hat{\Psi}}^{cs}) \leq d(\hat{\Psi}^n(\xi, p), \hat{\Psi}^n(\zeta, z)) + d(\hat{\Psi}^n(\zeta, z), \Gamma_{\hat{\Psi}}^{cs})$$

where $(\zeta, z) \in W_{loc}^{ss}((\xi, p); \hat{\Psi}) \cap (\Sigma_k \times D_{cs})$. According to Proposition 4.8 and (5.7) the right part of the above inequality converges to zero and therefore we conclude the claim. \square

To obtain the lemma we only need to note that $\mathcal{C}^+ \times \{y\} \subset \Sigma_{k,d}^- \times D_{cu}$ and thus by the above claim belongs to $W^u(\Gamma_{\hat{\Psi}}^{cu}; \hat{\Psi})$. On the other hand, using that $\hat{\Psi} \in \mathcal{V}$ and again the claim, it follows that

$$\hat{\Psi}^m(\mathcal{C}^+ \times \{y\}) \subset \Sigma_{k,d}^+ \times D_{cs} \subset W^s(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi}),$$

which concludes the proof of the lemma. \square

The next lemma shows the other intersection of the symbolic cycle, namely

$$W^u(\Gamma_{\Psi}^{cs}; \hat{\Psi}) \cap W^s(\Gamma_{\Psi}^{cu}; \hat{\Psi}) \neq \emptyset.$$

This is achieved by the following stronger result as a consequence of the blender properties.

Lemma 5.8. *It holds that $W^{uu}(\Gamma_{\Psi}^{cs}; \hat{\Psi}) \cap W^{ss}(\Gamma_{\Psi}^{cu}; \hat{\Psi}) \neq \emptyset$.*

Proof. Take the point $x \in B_{cs}$ from the cyclic intersection hypothesis satisfying $\phi_{k+1}^n(x) \in B_{cu}$. Since $B_{cu} \subset \mathcal{P}(\Gamma_{\Psi}^{cu})$, there exists a bi-sequence $\xi \in \Sigma_k$ such that $(\xi, \phi_{k+1}^n(x)) \in \Gamma_{\Psi}^{cu}$.

Consider $\zeta = (\zeta_i)_{i \in \mathbb{Z}} \in W_{loc}^s(\xi; \tau)$ with $\zeta_i = k+1$ for $-n \leq i < 0$, and let

$$z = \gamma_{\xi, \phi_{k+1}^n(x), \hat{\Psi}}^s(\zeta).$$

Remember that γ^s is the graph map of the local strong stable set, which is itself an almost δ -horizontal disk. Thus, $\|z - \phi_{k+1}^n(x)\| < \delta$ and by the hypothesis on the neighborhood \mathcal{V} of $\hat{\Phi}$ we have that

$$\hat{\Psi}^{-n}(\zeta, z) \in \Sigma_{k+2} \times B_{cs}.$$

By the property of the blender (see Proposition 4.9 and Remark 4.11), $W_{loc}^{ss}(\hat{\Psi}^{-n}(\zeta, z); \hat{\Psi}) \cap W_{loc}^{uu}(\Gamma_{\Psi}^{cs}; \hat{\Psi}) \neq \emptyset$. Let (β, t) be a point on this intersection, then it is in $W_{loc}^{uu}(\Gamma_{\Psi}^{cs}; \hat{\Psi})$ and since $(\beta, t) \in W_{loc}^{ss}(\hat{\Psi}^{-n}(\zeta, z); \hat{\Psi})$ then

$$\hat{\Psi}^n(\beta, t) \in W_{loc}^{ss}((\zeta, z); \hat{\Psi}) = W_{loc}^{ss}((\xi, \phi_{k+1}^n(x)); \hat{\Psi}) \subset W_{loc}^{ss}(\Gamma_{\Psi}^{cu}; \hat{\Psi}),$$

proving the lemma. \square

Combining the above two lemmas the proof of the proposition is now complete. \square

Now we are ready to prove Theorem B.

Proof of Theorem B. Fix positive constants, $\gamma, \hat{\gamma}, \lambda, \beta$ such that

$$1/2 < \gamma \leq \lambda < \beta < 1 \leq \hat{\gamma}^{-1} < 2.$$

We consider an arc of C^1 -diffeomorphisms $\{h_\varepsilon\}_{\varepsilon \in [0, \varepsilon_0]}$, $h_\varepsilon : M \rightarrow M$ such that $h_0 = \text{id}$ and for each $\varepsilon > 0$ the map h_ε has two hyperbolic fixed points $p = p(\varepsilon)$ and $q = q(\varepsilon)$ such that p is a sink, q is a source,

$$W^s(p, h_\varepsilon) \cap W^u(q, h_\varepsilon) \neq \emptyset,$$

and $d(p(\varepsilon), q(\varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We also assume that there are pairwise disjoint neighborhoods $D_{cs} = D_{cs}(\varepsilon)$ and $D_{cu} = D_{cu}(\varepsilon)$ of p and q such that the restrictions $h_\varepsilon : \overline{D_{cs}} \rightarrow D_{cs}$ and $h_\varepsilon^{-1} : \overline{D_{cu}} \rightarrow D_{cu}$ are (λ, β) -Lipschitz maps.

The next useful lemma tells us how to obtain the covering property from perturbations of a single map. Compare to [NP12, Prop. 3.2].

Lemma 5.9. *Let $\phi_1 : \overline{D} \rightarrow D$ be a (λ, β) -Lipschitz map with $\nu^\alpha < \lambda < \beta < 1$. Then there are a natural number k , an open neighborhood B of the fixed point of ϕ_1 and translations (in local coordinates) ϕ_2, \dots, ϕ_k of ϕ_1 such that $\overline{B} \subset \phi_1(B) \cup \dots \cup \phi_k(B)$. Consequently, the one-step map*

$$\Phi = \tau \times (\phi_1, \dots, \phi_k) \in \mathcal{S}_{k, \lambda, \beta}^{0, \alpha}(D)$$

has a symbolic cs-blender whose superposition domain contains B .

We observe that the number k of translations of ϕ_1 depends on the dimension of M and the contraction bound λ of ϕ_1 .

Proof of Lemma 5.9. This result is a consequence of Theorem A. Consider the open ball $B_\varepsilon(p) \subset D$ of radius $\varepsilon > 0$ centered at the fixed point p of ϕ_1 . Note that there are $k = k(\lambda, \dim M) > 0$ and points $d_1 = p$ and $d_i \in B_\varepsilon(p)$, $i = 2, \dots, k$, such that

$$\overline{B_\varepsilon(p)} \subset B_{\frac{\lambda}{2}\varepsilon}(d_1) \cup B_{\frac{\lambda}{2}\varepsilon}(d_2) \cup \dots \cup B_{\frac{\lambda}{2}\varepsilon}(d_k).$$

Now take (in local coordinates) the translations ϕ_i of ϕ_1 , $i = 2, \dots, k$, such that $B_{\lambda\varepsilon/2}(d_i) \subset \phi_i(B_\varepsilon(p))$. Then the choice of the points d_i and the inclusion above imply that $\overline{B_\varepsilon(p)} \subset \phi_1(B_\varepsilon(p)) \cup \dots \cup \phi_k(B_\varepsilon(p))$. Thus the one-step map $\Phi = \tau \times (\phi_1, \dots, \phi_k)$ satisfies Theorem A and therefore it has a symbolic cs -blender with $B_\varepsilon(p)$ contained in the superposition domain. \square

Going back to the proof of the theorem, by Lemma 5.9 there are k (depends only on λ and the dimension of M) and open sets $B_{cs} \subset D_{cs}$, $B_{cu} \subset D_{cu}$ containing p and q respectively such that

$$\overline{B_{cs}} \subset \phi_1^s(B_{cs}) \cup \dots \cup \phi_k^u(B_{cs}) \quad \text{and} \quad \overline{B_{cu}} \subset \phi_1^u(B_{cs}) \cup \dots \cup \phi_k^u(B_{cs})$$

where $\phi_1^s = h_\varepsilon$ on $\overline{B_{cs}}$, $\phi_1^u = h_\varepsilon^{-1}$ on $\overline{B_{cu}}$ and $\phi_i^s = \phi_i^s$, ϕ_i^u are, respectively, translations of ϕ_1^s , ϕ_1^u for all $i = 2, \dots, k$ (which depends on ε).

Set $\phi_1 = h_\varepsilon$ and let ϕ_i be $(\gamma, \hat{\gamma}^{-1})$ -Lipschitz C^1 -diffeomorphisms on M such that $\phi_i|_{\overline{B_{cs}}} = \phi_i^s$ and $\phi_i|_{\overline{B_{cu}}} = (\phi_i^u)^{-1}$ for all $i = 2, \dots, k$. Note that

$$\overline{B_{cs}} \subset \bigcup_{i=1}^k \phi_i(B_{cs}) \quad \text{and} \quad \overline{B_{cu}} \subset \bigcup_{i=1}^k \phi_i^{-1}(B_{cu})$$

and

$$\Phi = \tau \times (\phi_1, \dots, \phi_k) \in \mathcal{S}_{k, \lambda_{cs}, \beta_{cs}}^{1, \alpha}(D_{cs}) \cap \mathcal{S}_{k, \lambda_{cu}, \beta_{cu}}^{1, \alpha}(D_{cu})$$

with $\lambda_{cs} = \lambda$, $\beta_{cs} = \beta$, $\lambda_{cu} = \beta^{-1}$ and $\beta_{cu} = \lambda^{-1}$.

Now let us build a transition map between the regions D_{cs} and D_{cu} . Fix a small $\varepsilon > 0$ and consider the map, in local coordinates, $\varphi(x) = x + q - p$. From the choice of p and q , it holds $\varphi = \varphi_\varepsilon$ goes to the identity when $\varepsilon \rightarrow 0$.

By hypothesis, F has a Small horseshoe Λ . We consider a natural number ℓ such that $F^\ell|_\Lambda$ is conjugated to the full shift of d symbols with $d \geq k + 1$ and we take R_1, \dots, R_d rectangles in the ambient manifold N such that $\{R_1 \cap \Lambda, \dots, R_d \cap \Lambda\}$ is a Markov partition for $F^\ell|_\Lambda$. Moreover, assume that for every unit vectors v and w in the stable and unstable direction of Λ respectively it holds

$$\mu \leq \|DF^\ell(v)\| \leq \nu \quad \text{and} \quad \mu \leq \|DF^{-\ell}(w)\| \leq \nu$$

for some positive constant $\mu \leq \nu \leq \nu^\alpha < 1/2$ being $\alpha = \log \nu / \log \mu \in (0, 1]$.

We now modify $f_0 = F \times \text{id}$ in $f_0^{\ell-1}(R_i \times M)$ to get a one-parameter family of diffeomorphisms f_ε satisfying

$$\begin{aligned} f_\varepsilon^\ell|_{R_i \times M} &= F^\ell \times \phi_i \quad \text{for } i = 1, \dots, k \\ f_\varepsilon^\ell|_{R_i \times M} &= F^\ell \times \varphi \quad \text{for } i = k + 1, \dots, d. \end{aligned}$$

Note that the locally constant skew-product diffeomorphism f_ε^ℓ restricted to the set $\Lambda \times M$ is conjugated to the partially hyperbolic symbolic skew-product

$$\Psi_\varepsilon = \tau \ltimes (\phi_1, \dots, \phi_k, \varphi, \dots, \varphi) \in \mathcal{PHS}_d^{1,\alpha}(M).$$

In fact, from Proposition 5.2, for every small enough C^1 -perturbation g of f_ε , the iterate g^ℓ has an invariant set $\Delta \cong \Lambda \times M$ such that $g^\ell|_\Delta$ is conjugated with a $\mathcal{S}^{1,\alpha}$ -perturbation Ψ_{g^ℓ} of Ψ_ε .

Observe that Ψ_ε satisfies the assumptions in Proposition 5.5: the maps ϕ_i have the covering properties in the regions B_{cs}, B_{cu} , and the transition maps between these domains are created via the map φ and using the hypothesis that $W^s(p, h_\varepsilon) \cap W^u(q, h_\varepsilon) \neq \emptyset$. Then the symbolic skew-product Ψ_ε has a $\mathcal{S}^{1,\alpha}$ -robust symbolic cycle associated with symbolic blenders.

Hence, according to Proposition 5.2, using the conjugation, f_ε^ℓ has a C^1 -robust cycle associated with hyperbolic sets Γ_ε^{cs} and Γ_ε^{cu} for f_ε^ℓ that come from the symbolic blenders. To compute the co-index of this cycle note that the (stable) indices of Γ_ε^{cs} and Γ_ε^{cu} are, respectively, $\text{Ind}(\Lambda) + \dim(M)$ and $\text{Ind}(\Lambda)$. Thus, the co-index of the heterodimensional cycle is equal to the dimension of M .

The one-parameter family may be taken so that the same conclusions of a heterodimensional cycle with the above co-index between two hyperbolic sets can be made for the map f_ε . This ends the proof of the theorem. \square

5.3. Robust mixing: proof of Theorem C. We will build robustly mixing examples of symbolic skew-products. One of the classical ways to create robustly transitive diffeomorphisms is to construct a map that robustly has a hyperbolic periodic point with dense stable and unstable manifolds. Hence, using the inclination lemma (or λ -lemma) one concludes the diffeomorphism is robustly topologically mixing. We will do the same here for the symbolic case in the following theorem.

Theorem 5.10. *Let $\hat{\Phi} \in \mathcal{PHS}_d^{1,\alpha}(M)$ be a one-step map with $d > k$. Assume that the restriction*

$$\Phi = \hat{\Phi}|_{\Sigma_k \times M} = \tau \ltimes (\phi_1, \dots, \phi_k)$$

has a symbolic cs-blender $\Gamma_\Phi^{cs} \subset \Sigma_k \times M$ whose superposition domain contains an open set $B \subset M$ satisfying that $\overline{B} \subset \phi_1(B) \cup \dots \cup \phi_k(B)$.

Assume that there exists an attracting fixed point of ϕ_1 and a repelling fixed point of ϕ_{k+1} , both in B , having respectively stable and unstable manifold C^1 -robustly dense in M .

Then, there exists a fixed point $(\vartheta, p) \in \Gamma_\Phi^{cs}$ for $\hat{\Phi}$ such that its stable and unstable sets,

$$W^s((\vartheta, p); \hat{\Phi}) \quad \text{and} \quad W^u((\vartheta, p); \hat{\Phi}),$$

are $\mathcal{S}^{1,\alpha}$ -robustly dense on $\Sigma_d \times M$.

Consequently $\hat{\Phi}$ is $\mathcal{S}^{1,\alpha}$ -robustly topologically mixing.

The next sequence of propositions are necessarily to prove the above theorem. The first proposition shows how to enlarge the topological dimension of the closure of the strong unstable set of a blender via a fiber-repelling fixed point. Compare to [BDV05, Lemma 6.12].

Proposition 5.11 (Blender activation). *Let $\hat{\Phi} \in \mathcal{PHS}_d^{0,\alpha}(M)$ be a one-step map in the hypothesis of Proposition 4.9. Assume that there exists a fiber-repelling fixed point (v, q) of $\hat{\Phi}$ in $\Sigma_d \times B$. Then for every small enough $\mathcal{S}^{0,\alpha}$ -perturbation $\hat{\Psi}$ of $\hat{\Phi}$ it holds that*

$$W^u((v, q_{\hat{\Psi}}); \hat{\Psi}) \subset \overline{W^{uu}(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi})}$$

where $\Gamma_{\hat{\Psi}}^{cs}$ is continuation of $\Gamma_{\hat{\Phi}}^{cs}$ for $\Psi = \hat{\Psi}|_{\Sigma_k \times M}$.

Proof. Consider a small $\mathcal{S}^{0,\alpha}$ -neighborhood \mathcal{V} of $\hat{\Phi}$ such that for every $\hat{\Psi} \in \mathcal{V}$ $C\nu^\alpha < \delta$ where $C = C_{\hat{\Psi}}(1 - \gamma^{-1}\nu^\alpha)^{-1}$ and $\delta > 0$ is given in Proposition 4.9. Moreover, we can take δ small enough such that $B_{2\delta}(q) \subset B$.

Let $\hat{\Psi} \in \mathcal{V}$ and consider $(\xi, x) \in W^u((v, q_{\hat{\Psi}}); \hat{\Psi})$. Let V be an open neighborhood of (ξ, x) . In order to prove the proposition we need to show that the global strong unstable set of $\Gamma_{\hat{\Psi}}^{cs}$ meets V . Since $\Psi^{-n}(\xi, x)$ converges to $(v, q_{\hat{\Psi}}) \in \Sigma_d \times B$ then for n large enough we have that $B_\delta(\Psi^{-n}(\xi, x)) \subset \Sigma_k \times B$. Now, we define

$$h_n : W_{loc}^s(\tau^{-n}(\xi); \tau) \rightarrow M, \quad h_n(\zeta) = \mathcal{P} \circ \hat{\Psi}^{-n}(\tau^n(\zeta), x) = \psi_{\tau^{n-1}(\zeta)}^{-n}(x).$$

Let H_n be the graph set of h_n and notice that for n large

$$H_n \subset \Psi^{-n}((W_{loc}^s(\xi; \tau) \times \{x\}) \cap V).$$

Claim 5.12. *For n large enough, H_n is a δ -horizontal disk in $\Sigma_d \times B$.*

Proof. Let $\zeta, \zeta' \in W_{loc}^s(\tau^{-n}(\xi); \tau)$ be two bisquences with $d_{\Sigma_k}(\zeta, \zeta') = \nu^l$. Applying Lemma 4.2 to $\tau^n(\zeta), \tau^n(\zeta')$ and the point x , we obtain

$$\begin{aligned} \|h_n(\zeta) - h_n(\zeta')\| &= \|\psi_{\tau^{-1}(\tau^n(\zeta))}^{-n}(x) - \psi_{\tau^{-1}(\tau^n(\zeta'))}^{-n}(x)\| \\ &\leq C_{\Psi} \nu^{\alpha((n+l)-n)} \sum_{j=0}^{n-1} (\gamma^{-1} \nu^\alpha)^j \\ &< \nu^{\alpha l} C_{\hat{\Psi}} (1 - \gamma^{-1} \nu^\alpha)^{-1} = C d_{\Sigma_k}(\zeta, \zeta')^\alpha. \end{aligned}$$

Hence, from the choice of δ we have proved that h is (α, C) -Hölder continuous map with $C\nu^\alpha < \delta$. On the other hand, for n large enough $H_n \subset \Sigma_d \times B$ and therefore we conclude the proof of the claim. \square

From the above claim and Proposition 4.9, for n large enough,

$$W_{loc}^{uu}(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi}) \cap H_n \neq \emptyset.$$

This implies that

$$\emptyset \neq W^{uu}(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi}) \cap \Psi^n(H_n) \subset W^{uu}(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi}) \cap (W_{loc}^s(\xi; \tau) \times \{x\}) \cap V$$

and completes the proof of the proposition. \square

The following lemma shows how to construct a one-step map having fixed points with the stable or unstable set robustly dense.

Lemma 5.13. *Let $\Phi = \tau \ltimes \phi_\xi \in \mathcal{PHS}_d^{1,\alpha}(M)$ be a one-step skew-product map with a fiber-repelling/attracting fixed point (v, q) such that the unstable/stable manifold of q for ϕ_v is C^1 -robustly dense in M . Then the unstable/stable set of (v, q) for Φ is $\mathcal{S}^{1,\alpha}$ -robustly dense in $\Sigma_d \times M$.*

Proof. Consider a $\mathcal{S}^{1,\alpha}$ -perturbation $\hat{\Psi} = \tau \ltimes \psi_\xi$ of $\hat{\Phi}$. We will prove that the unstable set of the fiber-repelling fixed point $(v, q_{\hat{\Psi}})$, continuation of (v, p) for $\hat{\Psi}$, is dense in $\Sigma_d \times M$. Similarly follows for the case of the fiber-attracting fixed point and its stable set.

Let $\mathcal{C} \times V$ be any basic open set in $\Sigma_d \times M$. By the density of the unstable set of v for τ on Σ_d , there exists $\xi \in \mathcal{C} \cap W^u(v; \tau)$ and so for n large enough $\tau^{-n}(\xi)$ is close to v . For the fiber maps ψ_ζ with ζ close enough to v , there exists the fiber-repelling fixed point $q_{\hat{\Psi}, \zeta}$, which is the continuation of $q_{\hat{\Psi}}$.

Let \tilde{V} be the closure of a non-empty open set in V . Fixing n , observe that for every $m \geq 1$ it holds that

$$\psi_{\tau^{-1}(\xi)}^{-m-n}(\tilde{V}) = \psi_{\tau^{-m}(\tau^{-n}(\xi))}^{-1} \circ \cdots \circ \psi_{\tau^{-1}(\tau^{-n}(\xi))}^{-1} \circ \psi_{\tau^{-1}(\xi)}^{-n}(\tilde{V})$$

where $\psi_{\tau^{-i}(\tau^{-n}(\xi))}$ is C^1 -close to ψ_v and thus also to ϕ_v . Since by hypothesis the unstable manifold of q for ϕ_v is C^1 -robustly dense in M , the same holds for the fixed point $q_{\hat{\Psi}, \tau^{-i}(\tau^{-n}(\xi))}$ with respect to the maps $\psi_{\tau^{-i}(\tau^{-n}(\xi))}$.

By the continuous dependence of compact pieces of the unstable manifold with respect to the fiber diffeomorphisms, we may assume that for all $m \geq 1$, the unstable manifold of $q_{\hat{\Psi}, \tau^{-m}(\tau^{-n}(\xi))}$ intersects $\psi_{\tau^{-1}(\xi)}^{-n}(\tilde{V})$.

Thus for each $m \geq 1$ there is the point x_m which belongs to the intersection

$$\psi_{\tau^{-1}(\xi)}^{-n}(\tilde{V}) \cap W^u(q_{\hat{\Psi}, \tau^{-m}(\tau^{-n}(\xi))}; \psi_{\tau^{-m}(\tau^{-n}(\xi))}).$$

Since \tilde{V} is closed we may assume that the sequence $\{x_m\}$ converges to a point x in $\psi_{\tau^{-1}(\xi)}^{-n}(\tilde{V}) \subset \psi_{\tau^{-1}(\xi)}^{-n}(V)$. By the Hölder continuity of fiber maps on the base sequence, $q_{\hat{\Psi}, \tau^{-m}(\tau^{-n}(\xi))}$ tends to $q_{\hat{\Psi}}$, and then it is not hard to show that $\hat{\Psi}^{-m-n}(\tau^{-n}(\xi), x)$ goes to $(v, q_{\hat{\Psi}})$ as $m \rightarrow \infty$. Thus

$$\hat{\Psi}^n(\tau^{-n}(\xi), x) \in (\mathcal{C} \times V) \cap W^u((v, q_{\hat{\Psi}}); \hat{\Psi}),$$

concluding the proof of the lemma. \square

The following proposition demonstrates how to create a symbolic cs -blender with a dense strong unstable set.

Proposition 5.14. *Let $\hat{\Phi} \in \mathcal{PHS}_d^{1,\alpha}(M)$ be a one-step in the hypothesis of Proposition 4.9 with $d > k$. Assume that the fiber map ϕ_{k+1} has a repelling fixed point in B with the unstable manifold C^1 -robustly dense on M . Then for every small enough $\mathcal{S}^{1,\alpha}$ -perturbation $\hat{\Psi}$ of $\hat{\Phi}$ it holds that*

$$\overline{W^{uu}(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi})} = \Sigma_d \times M$$

where $\Gamma_{\hat{\Psi}}^{cs}$ is the continuation of $\Gamma_{\hat{\Phi}}^{cs}$ for $\Psi = \hat{\Psi}|_{\Sigma_k \times M}$.

Proof. Let $q \in B$ be the repelling fixed point of ϕ_{k+1} . Observe that (v, q) is a fiber-repelling fixed point of $\hat{\Phi}$ in $\Sigma_d \times B$ where $v = (v_i)_{i \in \mathbb{Z}} \in \Sigma_d$ is the bi-sequence with $v_i = k+1$ for all $i \in \mathbb{Z}$. By Proposition 5.11 it follows that for every small enough $\mathcal{S}^{1,\alpha}$ -perturbation $\hat{\Psi}$ of $\hat{\Phi}$,

$$W^u((v, q_{\hat{\Psi}}); \hat{\Psi}) \subset \overline{W^{uu}(\Gamma_{\hat{\Psi}}^{cs}; \hat{\Psi})}$$

By Lemma 5.13, $W^u((v, q_{\hat{\Psi}}); \hat{\Psi})$ is dense in $\Sigma_d \times M$ implying the density of the strong unstable set of $\Gamma_{\hat{\Psi}}^{cs}$ for $\hat{\Psi}$. \square

Lemma 5.15. *Consider a symbolic skew-product $\hat{\Phi} \in \mathcal{PHS}_d^{0,\alpha}(M)$ and let $\Gamma \subset \Sigma_d \times M$ be a $\hat{\Phi}$ -invariant set such that*

$$\Gamma \subset \overline{W^{uu}((\vartheta, p); \hat{\Phi})}$$

where (ϑ, p) is a fixed point of $\hat{\Phi}$. Then $W^{uu}(\Gamma; \hat{\Phi}) \subset \overline{W^{uu}((\vartheta, p); \hat{\Phi})}$.

Proof. Consider $(\xi, x) \in W^{uu}(\Gamma; \hat{\Phi})$ and let V be an open neighborhood of (ξ, x) . In order to prove the lemma we need to show that the strong unstable set of the fixed point (ϑ, p) meets V . Since (ξ, x) belongs to the global strong unstable set of Γ and Γ is $\hat{\Phi}$ -invariant set then there are $n \in \mathbb{N}$ and $(\zeta, z) \in \Gamma$ such that $\hat{\Phi}^{-n}(\xi, x) \in W_{loc}^{uu}((\zeta, z); \hat{\Phi})$. Hence

$$\hat{\Phi}^{-n}(V) \cap W_{loc}^{uu}((\zeta, z); \hat{\Phi}) \neq \emptyset. \quad (5.8)$$

Since the global strong unstable set of (ϑ, p) lies densely in Γ then there exists a sequence $((\xi^{(k)}, x^{(k)}))_k \subset W^{uu}((\vartheta, p); \hat{\Phi})$ converging to (ζ, z) .

According to Proposition 4.8, the local strong unstable set $W_{loc}^{uu}((\xi, x); \hat{\Phi})$ varies continuously with respect to (ξ, x) and so $W_{loc}^{uu}((\xi^{(k)}, x^{(k)}); \hat{\Phi})$ converges to $W_{loc}^{uu}((\zeta, z); \hat{\Phi})$. Since $\hat{\Phi}^{-n}(V)$ is an open set in $\Sigma_d \times M$, then this convergence and (5.8) imply that for k large enough

$$\hat{\Phi}^{-n}(V) \cap W_{loc}^{uu}((\xi^{(k)}, x^{(k)}); \hat{\Phi}) \neq \emptyset.$$

Therefore,

$$\emptyset \neq V \cap \hat{\Phi}^n(W_{loc}^{uu}((\xi^{(k)}, x^{(k)}); \hat{\Phi})) \subset V \cap W^{uu}((\vartheta, p); \hat{\Phi}),$$

completing the proof of the lemma. \square

Now, we are ready to give the proof of Theorem 5.10.

Proof of Theorem 5.10. Let us denote by (ϑ, p) and (v, q) , respectively, the fiber-attracting and fiber-repelling fixed points of $\hat{\Phi}$ in $\Sigma_d \times B$ where the bi-sequences $\vartheta = (\vartheta_i)_{i \in \mathbb{Z}}$ and $v = (v_i)_{i \in \mathbb{Z}}$ are given by $\vartheta_i = 1, v_i = k + 1$ for all $i \in \mathbb{Z}$. Observe that (ϑ, p) belongs to Γ_{Φ}^{cs} .

Since by hypothesis the stable manifold of p for ϕ_1 is C^1 -robustly dense on M , then we can apply Lemma 5.13 to conclude that the stable set of (ϑ, p) for $\hat{\Phi}$ is $\mathcal{S}^{1,\alpha}$ -robustly dense on $\Sigma_d \times M$.

Moreover, from Theorem 2.9, (see also Proposition 2.6), it holds that the strong unstable set lies $\mathcal{S}^{1,\alpha}$ -robustly dense in Γ_{Φ}^{cs} . Therefore, by Lemma 5.15 it follows that

$$W^{uu}(\Gamma_{\Phi}^{cs}; \hat{\Phi}) \subset \overline{W^{uu}((\vartheta, p); \hat{\Phi})}$$

in a $\mathcal{S}^{1,\alpha}$ -robust sense.

By Proposition 5.14, the strong unstable set of Γ_{Φ}^{cs} for $\hat{\Phi}$ is $\mathcal{S}^{1,\alpha}$ -robustly dense on $\Sigma_d \times M$ and thus we obtain that the unstable manifold of (ϑ, p) also lies $\mathcal{S}^{1,\alpha}$ -robustly dense in $\Sigma_d \times M$.

Therefore both the stable and unstable sets of (ϑ, p) are $\mathcal{S}^{1,\alpha}$ -robustly dense in $\Sigma_d \times M$.

In order to conclude the proof of the theorem we need to prove that $\hat{\Phi}$ is $\mathcal{S}^{1,\alpha}$ -robustly topologically mixing. Let $\hat{\Psi} = \tau \times \psi_{\xi}$ be a perturbation of $\hat{\Phi}$ and $(\vartheta, p_{\hat{\Psi}})$ the continuation of the fixed point (ϑ, p) . Consider a pair of

open sets \hat{U} and \hat{V} in $\Sigma_d \times M$. By the density of the stable and unstable sets of $(\vartheta, p_{\hat{\Psi}})$ for $\hat{\Psi}$, we can take two points

$$(\eta, y) \in W^s((\vartheta, p_{\hat{\Psi}}); \hat{\Psi}) \cap \hat{U} \quad \text{and} \quad (\zeta, z) \in W^u((\vartheta, p_{\hat{\Psi}}); \hat{\Psi}) \cap \hat{V}.$$

Fix $\varepsilon > 2\epsilon > 0$ small enough and assume that $\hat{\Psi}$ belongs to a small perturbative neighborhood of $\hat{\Phi}$ such that $C_{\hat{\Psi}}(1 - \gamma^{-1}\nu^\alpha)^{-1} < \epsilon$ and

$$W_{loc}^{uu}((\xi, x); \hat{\Psi}) \subset \Sigma_d \times B_\varepsilon(p_{\hat{\Psi}}) \quad \text{for all } (\xi, x) \in \Sigma_d \times B_\varepsilon(p_{\hat{\Psi}}). \quad (5.9)$$

Since (ζ, z) belongs to the unstable set of $(\vartheta, p_{\hat{\Psi}})$ and by Lemma 4.2, it follows that for every n large enough and $\xi \in W_{loc}^s(\zeta; \tau)$ with $d_{\Sigma_k}(\xi, \zeta) \leq \nu^n$ we have that

$$\begin{aligned} \|\psi_{\tau^{-1}(\xi)}^{-n}(z) - p_{\hat{\Psi}}\| &\leq \|\psi_{\tau^{-1}(\xi)}^{-n}(z) - \psi_{\tau^{-1}(\zeta)}^{-n}(z)\| + \|\psi_{\tau^{-1}(\zeta)}^{-n}(z) - p_{\hat{\Psi}}\| \\ &\leq C_{\hat{\Psi}}(1 - \gamma^{-1}\nu^\alpha)^{-1} + \epsilon < 2\epsilon < \varepsilon. \end{aligned}$$

By continuity, we can take an open set V in M and a cylinder C in Σ_d around of ζ of large enough length n such that $C \times V \subset \hat{V}$ and

$$W_{loc}^s(\tau^{-n}(\zeta); \tau) \times B_\varepsilon(p_{\hat{\Psi}}) \subset \hat{\Psi}^{-n}((W_{loc}^s(\zeta; \tau) \cap C) \times V) \subset \hat{\Psi}^{-n}(\hat{V}).$$

On the other hand, since (η, y) belongs to the stable set of $(\vartheta, p_{\hat{\Psi}})$ it follows that $\psi_\eta^m(y) \in B_\varepsilon(p_{\hat{\Psi}})$ for all m large enough. Hence, (5.9) implies that for every m large enough

$$\begin{aligned} \emptyset &\neq W_{loc}^{uu}(\hat{\Psi}(\eta, y); \hat{\Psi}) \cap (W_{loc}^s(\tau^{-n}(\zeta); \tau) \times B_\varepsilon(p_{\hat{\Psi}})) \\ &\subset \hat{\Psi}^m(W_{loc}^{uu}((\eta, y); \hat{\Psi}) \cap \hat{U}) \cap \hat{\Psi}^{-n}(\hat{V}). \end{aligned}$$

Therefore, for all n and m large enough $\hat{\Psi}^{n+m}(\hat{U}) \cap \hat{V} \neq \emptyset$, proving that $\hat{\Psi}$ is topologically mixing. \square

Observe that diffeomorphisms that have fixed points with robustly dense stable/unstable manifolds can be constructed on any manifold M by perturbations of time-one maps of gradient-like vector fields. This completes our objective in building robustly mixing examples in the set of symbolic skew-products (compare to [Hom12, Lemma 4.1] and the application in [HN11]). Lastly we conjugate these examples with partially hyperbolic sets on manifolds.

Proof of Theorem E. The construction of an arc of diffeomorphisms on $N \times M$ with robustly topologically mixing non-hyperbolic sets is similar to the proof of Theorem B. Thus we will just note the main differences.

As in Theorem B, we modify $f_0 = F \times \text{id}$ to get a one-parameter family of diffeomorphisms f_ε satisfying

$$\begin{aligned} f_\varepsilon^\ell|_{R_i \times M} &= F^\ell \times \phi_i \quad \text{for } i = 1, \dots, k \\ f_\varepsilon^\ell|_{R_i \times M} &= F^\ell \times \varphi \quad \text{for } i = k+1, \dots, d \end{aligned}$$

The maps $\phi_i = \phi_{i,\varepsilon}$ for $i = 2, \dots, k$ are obtained from Lemma 5.9 as small translations (on local coordinates) of a C^1 -perturbation $\phi_1 = \phi_{1,\varepsilon}$ of the identity map with an attracting fixed point p with unstable manifold C^1 -robustly dense in M .

Moreover, the iterated function system generated by ϕ_1, \dots, ϕ_k satisfies the covering property in an open set B_{cs} with $p \in B_{cs}$. The map $\varphi = \varphi_\varepsilon$

is a C^1 -perturbation of id with a repelling fixed point in B_{c_s} whose unstable manifold is C^1 -robustly dense on M .

Notice that the restriction of f_ε^ℓ to $\Lambda \times M$ is conjugated to

$$\Phi_\varepsilon = \tau \ltimes (\phi_1, \dots, \phi_k, \varphi, \dots, \varphi) \in \mathcal{PHS}_d^{1,\alpha}(M).$$

This partially hyperbolic one-step satisfies the assumptions in Theorem 5.10 and thus Φ_ε is $\mathcal{S}^{1,\alpha}$ -robustly topologically mixing. By Proposition 5.2, via the conjugation, it follows that $f_\varepsilon^\ell|_{\Lambda \times M}$ is C^1 -robustly topologically mixing and thus the same holds for $f_\varepsilon|_{\Lambda \times M}$.

To show that $\Lambda \times M$ and its continuations Δ_g (for small C^1 -perturbation g of f_ε) are non-hyperbolic sets, let us observe that the map Φ_ε by construction has fiber-repelling and fiber-attracting fixed points. Therefore, via the conjugation, the topologically mixing sets Δ_g on the manifold $N \times M$ will have periodic points of (stable) index $\text{Ind}(\Lambda) + \dim(M)$ and of index $\text{Ind}(\Lambda)$, thus proving the non-hyperbolicity of these sets.

This completes the proof of the last theorem. \square

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REFERENCES

- [AV10] A. Avila and M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. *Invent. Math.*, 181(1):115–189, 2010.
- [BD96] C. Bonatti and Lorenzo J. Díaz. Persistent nonhyperbolic transitive diffeomorphisms. *Ann. of Math. (2)*, 143(2):357–396, 1996.
- [BD08] C. Bonatti and L. J. Díaz. Robust heterodimensional cycles and C^1 -generic dynamics. *J. Inst. Math. Jussieu*, 7(3):469–525, 2008.
- [BD12] C. Bonatti and L. J. Díaz. Abundance of C^1 -homoclinic tangencies. *Trans. Amer. Math. Soc.*, 264:5111–5148, 2012.
- [BDV95] C. Bonatti, L. J. Díaz, and M. Viana. Discontinuity of the Hausdorff dimension of hyperbolic sets. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(6):713–718, 1995.
- [BDV05] C. Bonatti, L. J. Díaz, and M. Viana. *Dynamics beyond uniform hyperbolicity*, volume 102 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005.
- [BGMV03] C. Bonatti, X. Gómez-Mont, and M. Viana. Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(4):579–624, 2003.

- [BHN99] D. Broomhead, D. Hadjiloucas, and M. Nicol. Random and deterministic perturbation of a class of skew-product systems. *Dynam. Stability Systems*, 14(2):115–128, 1999.
- [Bon04] C. Bonatti. The global dynamics of C^1 -generic diffeomorphisms or flows. *conference in Second Latin American Congress of Mathematicians, Cancún, México*, 2004.
- [CP10] S. Crovisier and E. R. Pujals. Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms. *ArXiv e-prints*, November 2010.
- [Gor06] A. S. Gorodetskiĭ. Regularity of central leaves of partially hyperbolic sets and applications. *Izv. Ross. Akad. Nauk Ser. Mat.*, 70(6):19–44, 2006.
- [HN11] A. J. Homburg and M. Nassiri. Robust minimality of iterated function system with two generators. Preprint, 2011.
- [Hom12] A. J. Homburg. Circle diffeomorphisms forced by expanding circle maps. *Ergodic Theory and Dynamical Systems*, 32:2011–2024, 2012.
- [HPS77] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant manifolds*. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin, 1977.
- [Hut81] John E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.
- [IN12] Yu. Ilyashenko and A. Negut. Hölder properties of perturbed skew products and fubini regained. *Nonlinearity*, 25(8):2377, 2012.
- [New70] S. E. Newhouse. Nondensity of axiom A(a) on S^2 . In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, pages 191–202. Amer. Math. Soc., Providence, R.I., 1970.
- [NP12] M. Nassiri and E. R. Pujals. Robust transitivity in hamiltonian dynamics. *Ann. Sci. Éc. Norm. Supér.*, 45(2), 2012.
- [Pal00] J. Palis. A global view of dynamics and a conjecture on the denseness of finitude of attractors. *Astérisque*, (261):xiii–xiv, 335–347, 2000. Géométrie complexe et systèmes dynamiques (Orsay, 1995).
- [PS00] E. R. Pujals and M. Sambarino. Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. *Ann. of Math. (2)*, 151(3):961–1023, 2000.
- [PSW12] C. Pugh, M. Shub, and A. Wilkinson. Hölder foliations, revisited. *Journal of Modern Dynamics*, pages 79–120, December 2012.
- [RHRHTU07] F. Rodriguez-Hertz, M. A. Rodriguez-Hertz, A. Tahzibi, and R. Ures. A criterion for ergodicity of non-uniformly hyperbolic diffeomorphisms. *Electron. Res. Announc. Math. Sci.*, 14:74–81, 2007.
- [Sta99] J. Stark. Regularity of invariant graphs for forced systems. *Ergodic Theory Dynam. Systems*, 19(1):155–199, 1999.
- [Wil71] R. F. Williams. Composition of contractions. *Bol. Soc. Brasil. Mat.*, 2(2):55–59, 1971.

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